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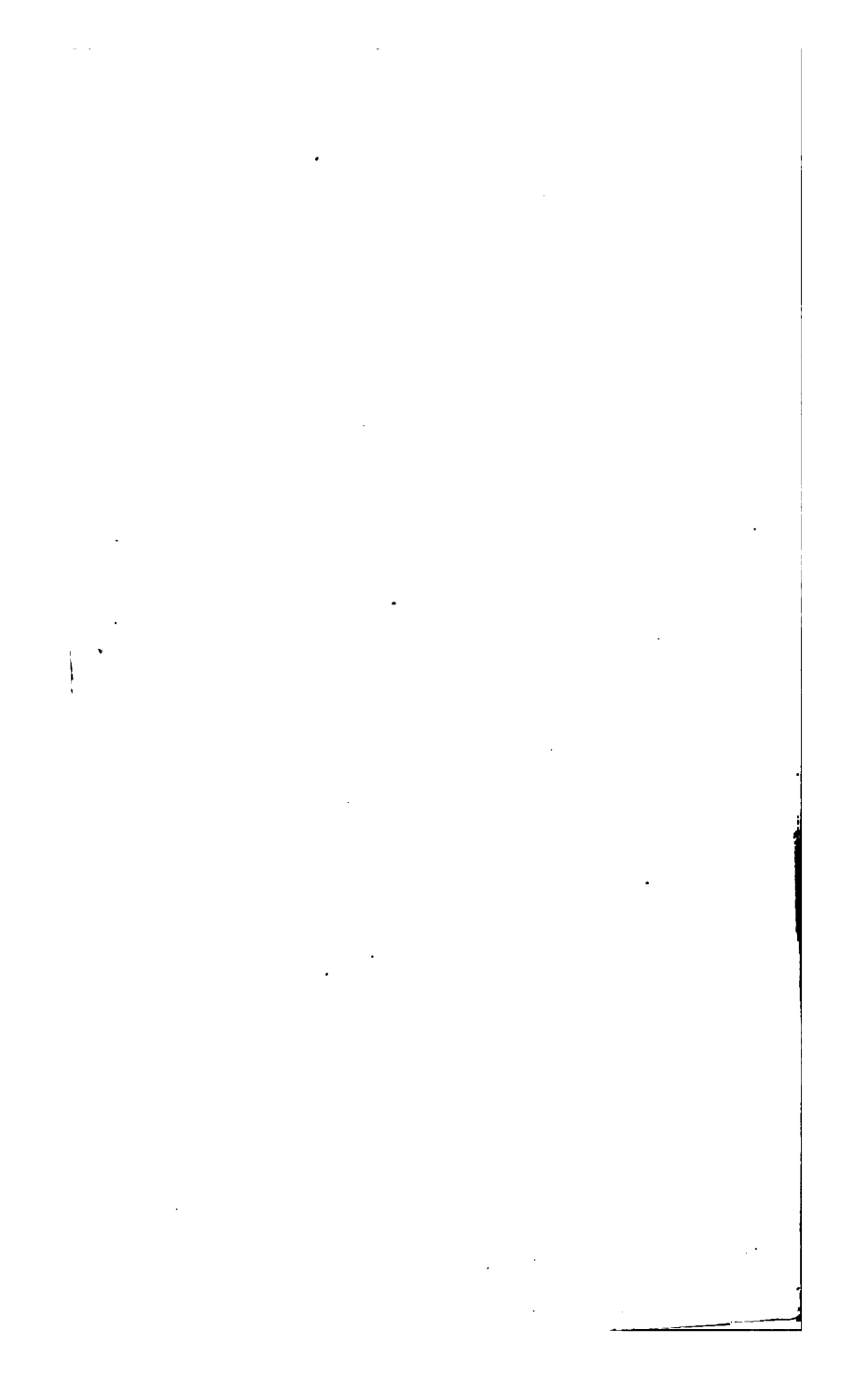
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AN EXPOSITORY SKETCH
OF A
NEW THEORY
OF THE
CALCULUS

BY
WILLIAM B. GREENE

PARIS
PRINTED FOR THE AUTHOR
140, AVENUE DES CHAMPS-ÉLYSÉES

JANUARY 1859



A NEW THEORY

OF THE

CALCULUS

THE GENEALOGY OF A

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NOTE

The writer attempts, in the following pages, to re-construct « THE CALCULUS » upon a basis altogether independent of considerations, not only of limits, but also of infinitely small elements. Giving to the term « differential » a meaning somewhat different from the one commonly received, and regarding the value of dx , or the differential of the independent variable, as constant, *and always equal to unity*, he endeavours to determine the formulas for all the correlative *finite values* which dy , or the differential of the dependent variable, may assume in the different cases where the equation $y = f(x)$ denotes an ordinary algebraic or transcendental functional relation, and also to show the utility of those formulas in the solution of problems. If it is acknowledged that he has done something (no matter how little) towards vindicating mathematical science from the reproach of demonstrating many of

its most important principles by arguments so fanciful and so arbitrary that « they would not be allowed in Theology », the measure of his ambition is filled. He permits himself to add that his method (which is presumed to be new) turns neither upon the relations which subsist between time, space and velocity, nor upon the possibility of the analytical development of values into series, but solely upon the facts of direction, curvature and tangency.

Persons desiring to obtain copies of this « Expository Sketch » (which is not for sale, and of which small edition only has been printed), will be so good as to send their names to the author immediately and to specify the way in which the pamphlet may be transmitted to them.

W. B. G.

PRELIMINARY PRINCIPLES.

Definitions.

1. Functions : An equation between two variables is an exact analytical statement of the particular form of the reciprocal dependance, as relating to value, which those variables have upon each other. Such an equation being given, as, for example, between the variables x and y , each variable is said to be *a function* of the other : that is, y is said to be a function of x , and x a function of y . When several variables are so connected with each other that every change which takes place in the value of any one of them, necessarily determines correlative changes in the values of all the others, and the relation is capable of being stated in an equation, then, each variable is said to be a function of all the others : thus, if $y = xz$, y is a function of x and z , x a function of y and z , and z is, etc.

Functions are either *increasing* or *decreasing*, *algebraic* or *transcendental*, etc.

2. When an increase in the value of one of the variables determines a correlative increase in the value of the other, the second is said to be *an increasing function* of the

first; and when an increase in the value of the one determines a correlative decrease in the value of the other, the second is said to be a *decreasing function* of the first. Thus, when $y = ax$, y is an increasing function of x ; but when $y = \frac{a}{x}$, y is a decreasing function of x .

3. When the equation that expresses the nature of the functional relation that subsists between x and y , is in one of the forms following :

$$y = a + x, y = a - x, y = ax, y = \frac{a}{x}, y = x^n, \text{ or } y = x^{\frac{m}{n}},$$

y is said to be an *algebraic function* of x ; but when the equation takes a form altogether different from any of those here cited, as, for example,

$$y = a^x, y = \log x, y = \sin x, \text{ etc.},$$

then y is said to be a transcendental function of x . Transcendental functions are, therefore, of various kinds, *exponential* functions, *logarithmic* functions, *circular* functions, etc.

4. If the functional relation that subsists between x and y is such that, when x takes, *continuously*, various values, y takes also, and by necessity, various *continuous* values that are strictly correlative to those taken by x , then y is said to be a *continuous function* of x .

5. **Direction.** The *direction* of a line is, at any one of its points, the particular manner according to which the line tends, at that special point, forwards, backwards or to one side. (An attempt to justify this definition of the term *direction*, is made, further on, in articles 13, 46 and 47.)

6. **Right lines and curves :** *A straight line* is one whose direction never changes. *A curve* is a line whose direction, as considered first at one point and then at another, varies always, but always in accordance with some rule; *curvature* being nothing other than a continuous and regular variation of direction. (See articles 43, 45 and 47; see also, for a better definition of curvature, art. 54.)

7. **Tangency.** When a right line and a curve have a common point, and the two lines possess, at this common point, the same direction, the two lines are said to be *tangent* to each other at the point they possess in common. (See article 18.)

8. **Convexity and concavity.** That side which a curve turns towards its tangent is called its *convex side*, and that side which it turns away from its tangent is called its *concave side* : convexity and concavity being nothing other than the progressive divergency which, on both sides of the point of tangency, exists between the variable direction of a curve and the constant direction of the right line that is tangent to it. (See figures 9, 13 and 44).

Axioms.

9. Two right lines, that have the same direction and a point in common, coincide throughout their whole lengths, and are not two right lines, but one only.

10. If a curve and a straight line have a common point, or, in general, if any two lines whatever have a common point, and different directions at their common point, they necessarily intersect each other at that point; and conversely, if they have a common point, but do not intersect

each other at that point, they have, at the point they possess in common, the same direction. (The apparent exception observable in the case of tangents to curves of contrary flexure will be noticed further on, in the third chapter of this Sketch.)

11. When y is a continuous function of x , the correlative and corresponding values of x and y are represented, and *adequately* represented, by the abscissas and ordinates of the points of the line which has for its equation the very equation which expresses the functional relation that subsists between x and y . And, conversely, etc.

12. When a curve is situated between its tangent and the axis of x , it turns towards the axis of x its concave side; but, on the contrary, when the tangent is situated between the curve and the axis of x , the curve turns its convex side towards the axis of x . (See figures 7 and 8.)

Observations on the foregoing articles.

13. Many geometers, rejecting the definition commonly given in the books, describe the straight line as one that lies evenly between its extreme points; thus showing that, in their opinion, the definition of the straight line requires, in order to its completeness, an express recognition of *the straight line's direction*. To say that a straight line is the shortest distance between two points, is to make an exact statement of fact; but such a mere statement of fact, however exact in itself, can never be accepted as a satisfactory definition of the straight line. We venture to affirm — and we take the liberty to put the reader on the stand as a witness to the truth of what we are about to say — that the

mind never really conceives of any line whatever, without, at the same time, conceiving of that line's direction. The inquiry, therefore, though perhaps curious, is not irrelevant to our subject, whether, when we have succeeded in eliminating from our conception of a line the element of its direction, we have not actually eliminated from our imagination the total conception of the line itself.

14. What clear conception (we permit ourselves to ask) is conveyed to the mind, either by the statement that a line is that which has length without breadth or thickness, or by the statement that a surface is that which has length and breadth without thickness? The mind cannot conceive — the testimony of the reader's consciousness being appealed to, as decisive in regard to the truth of the assertion — of a line as generated by the intersection of one length and breadth without thickness by another length and breadth without thickness, if it does not, at the same time, add, and from its own stock, to the notions of length, breadth and thickness, the further notion of *continuity*; the conception of continuity being absolutely essential to that of a line. We freely acknowledge that a line may be discontinuous, but remark that its discontinuity in some places implies its continuity in others, and that its discontinuity from beginning to end, that is, throughout its entire length, would be equivalent to its non-existence.

Length, breadth and depth are dimensions of space, and may be conceived of *as such*; but the conception of a line (especially of a curved line) involves something more than a mere abstract conception of length, or distance, since it is that which serves to mark out length, and along which length is measured. For, certainly, the conception of the distance that separates the two sides of a chasm, is some-

thing different from the conception of the line that joins them; although, if we are to conceive of the distance as measured, we must conceive of it as measured along some line : — along a straight line, if the shortest possible distance is to be taken, and, in other case, along some other special line, either curved or broken.

15. We do not pretend to decide whether the considerations upon which we are now occupied belong to the domain of natural science, or to that of pure metaphysics. We know not whether there are, or are not, any real lines in nature, but are inclined to believe that there are, although the artists, who ought to be competent in the matter, tell us there are none such. This, however, we know, that we are treating of the human faculty of forming distinct conceptions, which human faculty certainly constitutes a legitimate object of scientific observation. If our definition of a line, that it is that which has length, continuity and direction without either breadth or thickness, is too metaphysical, what shall we say of the received definition of the books, that a line is that which has mere abstract length, without breadth or thickness? For we suppose we are to understand by the expression, « metaphysical », that which is remote from concrete reality, and abstracted to the verge of vanishing tenuity.

16. We are fully aware of the difficulty the mind experiences in conceiving of curves as changing their directions at every one of their points. The conception of direction undeniably implies the conception of two distinct points, the point from which the direction obtains, and the point towards which the direction tends; and we acknowledge, nay, we affirm, that, in general, it is impossible to conceive

of both these points as situated in one and the same curve. Thus we not only acknowledge the force of the objection, considered as a mere statement of fact, but we appropriate it and present it as an argument in favour of our conception of curvature. To illustrate by an example : If we conceive the lower half of a circle to be cut off and taken away, we shall conceive of the extremities of the remaining upper semicircle as aiming, or pointing, directly downwards, towards points situated in the respective tangents drawn to the circle at the extremities so aiming or pointing, and those points situated at any distance from, *and out of*, the curve, which we may be pleased to assign.

17. Lines may be conceived as generated in various ways, as, for example, by the intersection of surfaces, or by the continuous motion of a point along a surface, etc.; but, let a line be originally generated in whatever manner it may, it can always, and with propriety, be regarded, after the act of generation is completed, as capable of becoming the path of a moving point. For, any line being given, we can certainly conceive of its course as followed, with more or less accuracy, by a point, and that point, the gross material point, if necessary, of a pencil. The attempt to rule out, by mere prescription, all consideration of lines as generated by the motion of points, is arbitrary, an encroachment on the inherent liberties of the human intelligence, and a thing to be resisted to the utterance. For, certainly, both observation and experience testify that lines are, occasionally, actually generated by the motion of points : as, for example, circles described by one foot of the dividers, the other foot remaining stationary; also, right lines described by the motion of a material point along the straight edge of a ruler. The path of a point so moving that it

tends always towards another point fixed in position, the tendency being never thwarted by any disturbing cause, is evidently a right line : and the direction of a right line may, therefore, be regarded as identical with the direction of the moving point of which that right line is the path. In like manner, the direction of a curve may be regarded as identical with the direction of the moving point of which that curve is the path : and, since the direction of such a moving point continually changes, we may say, and with perfect propriety, that a curve is a line whose direction continually changes. Whether the conceptions conveyed by these words are clear and distinct, and whether the statements made are accurate in point of fact, let the reader judge. In the case of the straight line, we have a moving point tending always towards, or from, another point which is distinct from the moving point and stationary ; both points being, however, in the same straight line. In the case of a curve, we have a varying direction determined by the shifting positions of two distinct points, both of which are in a constant motion determined in accordance with some fixed law, which is the law of the curve's formation and growth ; one of these points remaining always in the curve, since it is by its motion that the curve is marked out, while the other point remains always outside of the curve, since, by reason of its continual change of position, the curve always fails to reach it. It is thus that we explain, or, at least, attempt to explain, the meaning we attach to the word « direction », the term « continuity » requiring no explanation.

18. It is usual to conceive of the tangent to a curve, in one or the other of the two ways following : either (1) as a secant that has ceased to be a secant, by reason of its having been revolved around one of the points of secancy

until the second point of secancy fell into coincidence with the one that remained stationary, in which case the two points of secancy, by coalescing, became transformed into a single point of tangency; or (2) as a prolongation of one of the sides of the curve, the curve itself being regarded as really no curve, but a polygon of an infinite number of infinitely small sides. We trust the reader will acknowledge that our conception of the fact of tangency, as stated in article 7, has the merit (or perhaps the demerit) of being altogether foreign to the first of the conceptions here mentioned, and utterly exclusive of the latter.

Notation.

19. The expression, $y = f(x)$, signifies that y is a function of x , and is read : *y equal to a function of x*. The special forms of this general expression are written thus, $y' = f'(x)$, $y'' = f''(x)$, $Y = F(x)$, etc., in order to distinguish from each other the particular cases of functional relation that may present themselves.

20. Let obv (fig. 4) be any curve whatever, referred to the rectangular coordinate axes X and Y , and having for its equation, $y = f(x)$. At any point, b , of the curve, taken at pleasure, suppose the tangent abg to be drawn : then, since b may be any point whatever of the curve, the coordinates of the point of tangency will be x and y . Through this point, whose coordinates are x and y , draw the line bh , parallel to the axis of x . Finally, on the line bh , and at a convenient distance from b , *assumed arbitrarily*, draw the perpendicular hk . Then will the distance bh , thus arbitrarily assumed, be called, at least in these pages, *the differential of x*, and designated thus :

dx . In like manner the distance hk , intercepted between the line bh and the tangent ag , will be called, at least in these pages, *the differential of y* , and designated thus : dy .

21. The differential of y is thus nothing other than *the difference between the ordinates bp and kq (not of the curve, but) of the tangent to the curve ag* ; and the differential of x is nothing other than the difference of the abscissas op and oq .

22. Except in those cases where the reader will be expressly warned of the contrary, *we shall always*, in these pages, *assume the differential of x as a constant quantity, and equal to one French centimetre*; and we shall also adopt that special length as the unity of linear measure. We assume, therefore (art. 20), $dx = \text{unity}$.

Essential nature of the differential fraction, $\frac{dy}{dx}$.

23. By reason of the nature of the differential triangle (art. 20), we have the following proportion always given :

$$dy : dx :: \text{tang. of ang. } bap : \text{unity},$$

$$\text{or, } \frac{dy}{dx} = \text{tang. of ang. } bap.$$

Therefore, *the differential of y , divided by the differential of x , is always equal to the trigonometrical tangent of the angle which the tangent to the curve, drawn through the point whose coordinates are x and y , makes with the axis of x .*

[We might easily demonstrate, at this place, the properties of similar triangles, basing our demonstration on the

principle of directions : but we apprehend that such a demonstration is entirely uncalled for.]

24. Moreover (art. 20), $dy : dx :: y : \text{subtang.}$;

$$\text{therefore, } \frac{dy}{dx} = \frac{y}{\text{subtang.}}$$

25. [It is evident that, in order to the utility of either of these propositions, the exact determination of dy (dx being assumed equal to unity) is necessary as an indispensable prerequisite. But we request the reader to look at the heading of this chapter, « Preliminary principles », and to feel assured that an attempt to explain the method for obtaining the exact determination of dy will be made in a subsequent chapter.]

26. At the extremities of the abscissas x , $x + dx$, $x + 2dx$, etc., let ordinates be erected, and let those ordinates be prolonged, if necessary, until they cut the tangent which is drawn to the curve through the special point whose coordinates are x and y . The curve and the tangent so drawn have, at their common point, that is, at the point of tangency, whose abscissa is x , the same ordinate, y ; but the ordinate of the tangent, taken for the abscissa $x + dx$, will be (ordinarily) greater or smaller than the corresponding ordinate of the curve : greater, if the curve turns its concave side, and smaller if it turns its convex side to the axis of x (art. 8). Thus, the constant increment (or, perhaps, decrement) of x being dx , or unity, the increment (or decrement) of the ordinate of the tangent, or dy (art. 24) will also be constant; for the rate of increase of the ordinate of the tangent is always the same, wherever it may be measured, since the tangent is a special right

line, with a special direction, and cutting the axis of x (if it cuts it at all) at a special angle : but the increment (or, perhaps, decrement) *of the ordinate of the curve* will vary regularly with the variations of x , because, etc. (arts. 6 and 8), and the increment of the ordinate of the curve will not (except in certain special and extraordinary cases) be equal to dy .

It is here assumed, *as self-evident*, that the ordinate of a straight line having a special direction, and cutting the axis of x at a special definite angle, has a constant rate of increase, equal (dx and the trigonometrical radius being taken each equal to unity) to the trigonometrical tangent of the angle at which that line cuts the axis of x .

But since (art. 7) the direction of a curve (at the special point of tangency whose abscissa is x) is identical with that of the tangent drawn to it at that point, it follows that the variable rate of increase of the ordinate of the curve (or of its decrease, since the increase may be positive or negative), has a special value at the point whose abscissa is x . which special value is identical with the constant value of the rate of increase of the ordinate of the tangent.

Or, in other words : *A line and its tangent, being two lines having a common point, possess, by consequence, a common ordinate at that common point; and since they have, at that common point, the same direction, they are, at that point, though perhaps at that point only, identical, and their ordinates possess, at that point, though perhaps at that point only, the same rate of increase.*

Therefore, *the rate of increase of the ordinate of a curve, and the rate of increase of the ordinate of its tangent, are, at the point of tangency, identical, and (x and y being the coordinates of the point of tangency)*

equal to $\frac{dy}{dx}$ (art. 23) : which, indeed, is self-evident, and is perhaps obscured, instead of being rendered plainer, by the effort here made to bring it into light.

Constant quantities have no differentials.

27. For the very reason that a constant quantity has no variation whatever, it has no rule, law or *rate* of variation : for that which, from its nature, is incapable of having any being, has no law or rule of existence. The variation of a constant is, therefore, zero, and its differential is also, of necessity, zero.

28. If we have given a curve whose equation is $y = f(x) \pm a$, nothing can prevent us from drawing a new axis of x , parallel to the one first given, and at a distance above or below it (as the case may be) equal to $\pm a$; and then the equation of the curve will take the form (as referred to the new axes) of $Y = f(x)$. We observe, first, that any point of the curve, having for its coordinates x and y , will be identical with the point which has for its coordinates x and Y ; and, secondly, that, since the new axis of x has the same direction with the old one, the straight line that is drawn through the point (x, y) , parallel to the old axis, will be identical with the straight line drawn parallel to the new axis through the point (x, Y) . Thus, since dx is always and everywhere equal to unity, and as the direction of the tangent to the curve is independent of the position of the origin of coordinates, the differential triangle will be one and the same thing for both of the systems of coordinates, and we shall have, by consequence, $dY = dy$, or $d[f(x) \pm a] = d[f(x)]$. Consequently, if we differen-

tiate any equation of the form $y = f(x) \pm a$, the constant, a , disappears in the act of differentiation, without leaving any trace whatever of its existence.

DIFFERENTIATION OF ALGEBRAIC FUNCTIONS.

Differentiation of functions of the form, $F(x) = n \times f(x)$.

29. Let abc (fig. 2) be any line whatever that has for its equation $y = f(x)$, and $a'de$ a line that has for its equation $Y = n \times f(x)$.

Let the point b , whose special coordinates are x and y , be any particular point whatever of the line abc ; and let d , whose special coordinates are x and Y , be the corresponding point (as taken for the same value of the abscissa) of the line $a'de$.

The ordinate Y , which is a special ordinate, since its corresponding abscissa has a special value, x , is at once the ordinate of the line $a'de$, and the ordinate of the tangent that is drawn to $a'de$, at d .

Moreover, Y , regarded as ordinate of the line $a'de$, has (or may have) a variable rate of increase; while the same ordinate, Y , regarded as ordinate of the straight line that is drawn tangent to $a'de$, at d , has a constant rate of increase. Therefore, Y , which is identical to itself, as having (the abscissa being x) the special and determinate value Y , differs from itself, inasmuch as it concentrates, in one single embodiment, two separate natures, the nature of the ordinate of a special line, and the nature of the ordinate of a special tangent to that line.

30. Y , regarded as a special ordinate to the line $a'de$, is equal to $n \times y$; but, when we say, "equal to $n \times y$ ",

we do not give an adequate expression to the exact fact of the case; for the relation is not one of equality, but one of identity : Y is $n \times y$. Y contains y exactly n times, and $n \times y$ is the very substance, or, if we may be allowed so strong an expression, the matter itself which composes Y . Consequently, Y , which contains y , n times, and is nothing other than n times y , increases with the increase itself of y , and not with some other increase which is a mere effect of that of y . Moreover, since it is the actual increase itself of y , n times repeated (and not some other increase merely equal to that of y), which constitutes the increase of Y , the increase of y enters, as often as it does enter, into that of Y , with all its essential and inseparable characteristics, attributes and qualities : therefore, *the rate of increase of y repeats itself n times in the rate of increase of Y* ; and we have, consequently, the rate of increase of Y equal to n times the rate of increase of y . But (art. 26) the rate of increase of Y is (the abscissa having the special determinate value, x) $\frac{dY}{dx}$, and the rate of increase of y is (the abscissa receiving the same special determination) $\frac{dy}{dx}$. We have, therefore,

$$\frac{dY}{dx} = n \times \frac{dy}{dx};$$

which is equivalent to saying that, since Y is *always n times y* , if y increases with *any given rate* of rapidity, Y will increase with *a rate of rapidity exactly n times as great* : which statement may safely be allowed to make its own way, as perfectly self-evident.

31. But the special rate of increase which the ordinate Y ,

of the line $a'de$, has at d , is identical (art. 26) with the constant rate of increase which the ordinate of the tangent that is drawn to the line $a'de$, at d , has always, let it be measured where it may. Therefore, when x becomes $x + dx$ (the abscissa of the point of tangency retaining always its determinate value x), y becomes $y + dy$, and Y becomes $Y + dY$; $y + dy$ and $Y + dY$ being ordinates to the tangents that are drawn to the curves abe and $a'de$, at the points b and d , which have the common abscissa x , but ordinates taken for a common abscissa, $x + dx$. Therefore (art. 24),

$$dY = n \times dy.$$

32. This theorem, though speculatively true, is, of course, of no practical utility whatever so long as the value of dy remains undetermined. Fortunately, however, in the only case that is of vital importance, the value of the dy presents itself spontaneously. Making $f(x) = x$, and thus giving to the equation $y = f(x)$ its most simple form, we obtain $y = x$, or the equation of a right line that cuts the axis of x at an angle of 45° . In this particular case, dy is always equal to dx , since y and x are identical as respects value, and differ as respects position only. Besides, the line whose equation is $y = x$, is always its own tangent.

Substituting, therefore, in the equation, $d(n \times y) = n \times dy$, for y and dy , the values x and dx , we obtain,

$$d(n \times x) = n \times dx.$$

33. Since $d(-y)$ is equal to the differential of the product formed to the multiplication of y by -1 , we have

$$\begin{aligned} d(-y) &= -dy, \text{ and also, for a similar reason,} \\ d(-x) &= -dx. \end{aligned}$$

Differentiation of functions of the form,

$$F'(x) = f(x) + F(x).$$

34. Let *abc* (fig. 3) be any line whatever which has for its equation $y = f(x)$; *a'de* any line whatever which has for its equation $Y = F(x)$; and *gh* the line that has for its equation $U = y + Y$.

Let the point *b*, whose coordinates are x and y , be any special point whatever of the line *abc*; let *d*, whose coordinates are x and Y , be the corresponding point (as taken for the same abscissa) of the line *a'de*, and let *g*, whose coordinates are x and U , be the corresponding point of the line *gh*.

The ordinate U , which is a special ordinate, since its corresponding abscissa has a special value, x , is at once the ordinate of the line *gh*, and the ordinate of the tangent that is drawn to that line, at *g*. U has, therefore, a two-fold nature, one nature as ordinate of the line *gh*, and as possessing a consequent variable rate of increase, and another nature as ordinate of the tangent drawn to *gh*, at *g*, and as possessing a consequent constant rate of increase.

35. U , regarded as the ordinate of the line *gh*, is, identically, $y + Y$; $y + Y$ existing in U with all their essential and inseparable attributes, characteristics and qualities. Consequently, if (the abscissa having the special determinate value x), y has a rate of increase, $\frac{dy}{dx}$, and Y a rate of increase, $\frac{dY}{dx}$, U has a rate of increase,

$$\frac{d(y + Y)}{dx} = \frac{dy}{dx} + \frac{dY}{dx}$$

But the special rate of increase which the ordinate U ,

of the line gh, has, at g, is identical (art. 26) with the constant rate of increase which the ordinate of the tangent that is drawn to gh, at g, has always, let it be measured where it may. Therefore, when x becomes $x + dx$ (the abscissa of the point of tangency retaining always its determinate value x), y becomes $y + dy$, Y becomes $Y + dY$, and U becomes $U + dU$; $y + dy$, $Y + dY$ and $U + dU$ being ordinates to the tangents that are drawn to the curves abe, a'de and gh, at the points b, d and g, which have the common abscissa x , but ordinates taken for a common abscissa, $x + dx$. Therefore, (art. 24)

$$dU = d(y + Y) = dy + dY,$$

and $dY = d(U - y) = dU - dy.$

36. To make these theorems of practical utility, we must know either the values of dy and dY , or those of dy and dU . Thus, if $y = nx$ and $U = mx$, we have $dY = d(U - y) = d(mx) - d(nx) = m dx - n dx = (m - n) dx$ (art. 34); but, if $y = \frac{a}{x}$ and $Y = x^n$, we shall have $dU = d\left(\frac{a}{x}\right) + d(x^n)$, and the values of $d\left(\frac{a}{x}\right)$ and $d(x^n)$ will have to be found by processes not yet explained.

Differentiation of functions of the form,

$$F'(x) = f(x) \times F(x).$$

37. Let abc (fig. 4) be any line whatever which has for its equation $y = f(x)$; a'de any line whatever that has for its equation $Y = F(x)$; and gh the line that has for its equation $U = y \times Y$.

Let the point b, whose coordinates are x and y , be any special point whatever of the line abc; let d, whose coordinates are x and Y , be the corresponding point (as taken

for the same value of the abscissa) of the line $a'de$; and let g , whose coordinates are x and U , be the corresponding point of the line gh .

The ordinate U , which is a special ordinate (since its corresponding abscissa has a special value, x), is at once the ordinate of the line gh , and the ordinate of the tangent that is drawn to gh , at g .

U , regarded as a special ordinate to the line whose equation is $U = y \times Y$, has a determinate value, which is, identically, $y \times Y$; and also a special rate of increase formed out of the special rates of increase which y and Y have when the abscissa receives the determinate value, x . For, since U is, in its identical substance, $y \times Y$, and, regarded as the ordinate of the line gh , never anything other than $y \times Y$, U grows *with the growths themselves* of y and Y , and not otherwise. But U grows with the growth of y as many times (and no more than as many times) as y enters, in its capacity of a growing function, into the ordinate of the point of tangency, g ; that is, into the product already formed by a multiplication already accomplished of y by Y . U grows, therefore, with the growth of y , Y times, and no more; and grows with the growth of Y , y times, and no more; for y enters Y times into the ordinate of the point of tangency, and no more, and Y , y times, and no more. Therefore, since the growth of U is nothing other than an exact repetition, under a form of totality, of all the particular growths of y and Y , those particular growths of y and Y , as occurring in y and Y , being identical with themselves as occurring in the total growth of U , the difference being one, not of values, nor even of substance, but solely of the localities where identical values are counted, it follows that the rate of increase of y enters into the rate of increase of U , Y times, or as many times

as y is contained in U ; and, also, that the rate of increase of Y enters into the rate of increase of U , y times, or as many times as Y itself is contained in U . Therefore,

$$\frac{dU}{dx} = y \times \frac{dY}{dx} + Y \times \frac{dy}{dx}.$$

But the special rate of increase, which the ordinate U of the line gh has at g , is identical (art. 26) with the constant rate of increase which the ordinate of the tangent that is drawn to the line gh at g , has always, be the value of x what it may. Therefore, when x becomes $x + dx$ (the abscissa of the point of tangency retaining always its determinate value, x), y becomes $y + dy$, Y becomes $Y + dY$, and U becomes $U + (y \times dY) + (Y \times dy)$; $y + dy$, $Y + dY$, and $U + (y \times dY) + (Y \times dy)$, being ordinates to the tangents that are drawn to the three curves abc , $a'de$ and gh , at the points b , d and g , which have the common abscissa x , but ordinates taken for a common abscissa, $x + dx$. Therefore (art. 24),

$$dU = (y \times dY) + (Y \times dy).$$

38. The ordinate of the curve, whose equation is $U = y \times Y$, is, when taken for the abscissa $x + dx$,

$$(y + \text{increment of } y) \times (Y + \text{increment of } Y) = (y \times Y) + (y \times \text{inc. } Y) + (Y \times \text{inc. } y) + (\text{inc. } y \times \text{inc. } Y),$$

a value which we have, as yet, no means of determining, but one which, in general, *differs* from that of the ordinate to the tangent taken for the same abscissa $x + dx$, for the reasons stated in articles 6 and 8.

39. Putting $Y = uz$, we have $U = y \times uz$, and $dU = d(yuz) = uz \times dy + y \times d(uz)$; therefore, $d(yuz) = (yu \times dz) + (yz \times du) + (uz \times dy)$.

40. Putting $U = a$, in which case U becomes the ordinate of a right line drawn parallel to the axis of x , we have $a = y \times Y$, or $Y = \frac{a}{y}$, and $dY = d\left(\frac{a}{y}\right)$. Differentiating both sides of equation $a = y \times Y$, we obtain, $0 = ydY + Ydy$, or $dY = -\frac{Ydy}{y}$, or, substituting for Y its value $\frac{a}{y}$, $d\left(\frac{a}{y}\right) = -\frac{ady}{y^2}$. We have, therefore, $d\left(\frac{a}{f(x)}\right) = -\frac{a \times df(x)}{[f(x)]^2}$; and, for the special case where $f(x) = x$, $d\left(\frac{a}{x}\right) = -\frac{adx}{x^2}$.

41. Putting $y = x$, and $Y = x$, we have $U = x^2$, and $dU = d(x^2)$. Differentiating both sides of equation $U = x \times x$, we obtain $dU = xdx + xdx = 2xdx$. Putting $y = x$, and $Y = x^2$, we have, $U = x \times x^2$, and $dU = x \times d(x^2) + x^2(dx) = 3x^2dx$, since $x \times d(x^2) = x \times 2xdx$. In the same way we find, $d(x^4) = 4x^3dx$, $d(x^5) = 5x^4dx$, and, in general, n being any positive whole number whatever,

$$d(x^n) = nx^{n-1}dx.$$

The differential of x^{-n} , that is of $\frac{1}{x^n}$, is found, by means of the process indicated in article 40, to be,

$$-nx^{-n-1}dx.$$

To find the differential of $x^{\frac{n}{m}}$, put $x^{\frac{n}{m}} = q$. Then, by differentiating both sides of the equation, $x^n = q^m$, and afterwards replacing q by its value, $x^{\frac{n}{m}}$, we obtain,

$$d\left(x^{\frac{n}{m}}\right) = \frac{n}{m} x^{\frac{n}{m}-1} dx.$$

Therefore, n being any number whatever, entire or fractional, positive or negative, we have always,

$$d(x^n) = nx^{n-1}dx.$$

42. We are now enabled to differentiate the algebraic functions that are mentioned, as such, in article 3. The differential of $(a + x)$ is (art. 28) dx ; that of $(a - x)$ is $-dx$; that of ax is (art. 32) adx ; that of $\frac{a}{x}$ is (art. 40)

$-\frac{adx}{x^2}$, and that of x^n , n being either positive or negative,

entire or fractional, is $nx^{n-1}dx$. Moreover, since the method followed in our investigation has been exceedingly general, we may substitute, instead of x , in any one of these formulas, any function of x , however complicated that we please, and the truth of the formula will not be in any way affected. For example, it would be easy to show (by art. 44) that $d([f(x)]^n) = n[f(x)]^{n-1} \times d[f(x)]$; and so of the other cases. We are tempted to speak here of partial differentiation, but we forbear; for, on one side, our subject is infinite, and, on the other, it is necessary that we should confine ourselves within the limits of a simple outline, in order that our exposition may not lose itself in the multiplicity of details. We request the reader, however, to believe, with the second hero of Niagara (notwithstanding the ominous character of his language) that « some things may be done as well as others ».

Observations.

43. The reasoning employed (art. 30, 34, 32) to show that $d(nx)$ is equal to $n \times dx$ appears to be of the kind which leaves no doubt or fluctuation in the mind. If, therefore, we conceive x , considered as a continuous linear value, having a rate of increase, to be multiplied by another x , considered as a numerical quantity of equal value, we shall have no difficulty in conceiving of $d(x \times x)$ as equal to xdx , plus something; but when, in order to complete the expression, we invert our conception, and endeavour to imagine the numerical x as possessing a continuous rate of increase, and as being multiplied by the linear x , we find ourselves thrown into a state of intellectual impotency that is simply ludicrous.

For, first, it is as difficult to conceive of a number multiplied by a line, as it is easy to conceive of a line multiplied by a number; and, secondly, the notion of a number as possessing a continuous rate of increase cannot be formed by the human mind. Number is necessarily conceived as discontinuous. The expression, « a continuous number », is absurd.

It is easy to conceive of an *actually* discontinuous line; for the centimetre which we take off from the scale in our dividers, to serve as an actual unity of measure, is an example in point; since the centimetre, so taken, exists, as a line, in its extremities only, that is in the two extreme points of the dividers. Nevertheless, *virtually*, the centimetre, even when so taken, is continuous, and the mind conceives it as such. But when a line increases by the addition of one such centimetre to another, it increases discontinuously, and by jumps of one centimetre at a time. The mind finds no diffi-

culty in conceiving a line as being formed by the continual accretion of equal lines that already exist; but it possesses no faculty whose function it is to conceive of number as continuous. There is no melting of one number into another. When a number becomes greater or smaller, it ceases to be the number it was, and becomes another number by a jump. Each number is separated from every other by a discontinuous chasm. We may lighten the task of our imagination by supposing this chasm to be divided into a great many little fractional parts; but we only succeed, by so doing, in substituting many small chasms in the stead of one great one. A number may increase by having added to it the numerical unit, or a very small fraction of that unit; but, in either case, it increases by a jump commensurate with the value of the quantity added. We conclude, therefore, that the method of measuring actual lines numerically is perfectly legitimate, provided, always, that we remember we are measuring what we do measure, length or distance, and that we are not measuring continuity, which is something altogether foreign to the nature of numerical measure.

44. An equation being given, $U = y \times Y$, y and Y being both variables and functions of x , we conceive of U , y , Y , as three ordinates, each of which contains the centimetre, or unit of length, a certain number of times; the number of U being always equal to the number of y multiplied by the number of Y . When, therefore, x receives a definite numerical value, as containing a special number of centimetres, then y and Y take special correlative numerical values, and, consequently U takes also a special correlative numerical value. In determining the three ordinates corresponding to a special value of x , we have, therefore, nothing

whatever to do with considerations of continuity, since we are considering, not constant values which remain *always* the same, nor variable values which are *always* changing, but special values, to which the considerations which lie at the foundation of the conception of increase or decrease, are utterly foreign. For, if a constant value may be considered as one that « stands », and a variable value as one that « flows », a special value must be considered as a flowing value that has been absolutely deprived of its flowing character. For a variable varies with the variation of its circumstances, and a constant remains the same amid the variation of its circumstances, while a special value is one placed in unvarying circumstances.

Moreover, the rates of increase of y and Y , and consequently of U , are, when x takes the special determination, x , neither variable rates, nor constant rates, but special rates : and since y enters into U , by the conditions of the question, not simply as the special value y , but as that special value having a special rate of increase ; and since Y enters into U , not simply as the special value Y , but as that special value having also a special rate of increase — y and Y being given, by the conditions of the question, *as variables* having determinate rates of increase ; it follows that U has a special rate of increase equal to the rate of increase of the ordinate y , regarded as such and taken as many times as the ordinate y is numerically contained in U , that is Y times, plus the rate of increase of the ordinate Y , regarded as such and taken as many times as the ordinate Y is numerically contained in U , that is y times. For the rate of increase of the ordinate U is identically composed of the rates of increase of the ordinates y and Y , the three ordinates being regarded as having continuous rates of increase, but continuous rates of increase which take

special values when x takes the special determination, x . Therefore, the rate of increase of U is equal to the rate of increase of y , multiplied by the numerical value Y , plus the rate of increase of Y , multiplied by the numerical value y ; y , Y and U being simultaneously considered as ordinates increasing continuously, and as numbers increasing discontinuously, the discrepancy between these two opposing and contradictory points of view being altogether obviated and annulled by the consideration of the ordinates as special ordinates, and of their rates of increase as special rates of increase, the abscissa being supposed to have received the special determination, x centimetres.

45. Whether these explanations appear clear, or the contrary, we know not; but we request the reader to suspend his judgment until we shall have an opportunity to state (in the next chapter) our theory of curvature — a theory altogether independent (nay, utterly exclusive) of the hypothesis that a curve is composed of rectilinear elements.

46. We take the liberty to illustrate the foregoing observations by a particular example. Let $y = \sqrt{x}$. $Y = 2y = 2\sqrt{x}$, and $U = y \times Y = 2x$ (fig. 46). When x receives the special determination, $x = 4$ centimetres, we shall have $y = 2$ centimetres, $Y = 4$ centimetres, and $U = 8$ centimetres.

Also, since $dy = \frac{1}{2} x^{-\frac{1}{2}} dx$ (art. 41), $dY = x^{-\frac{1}{2}} dx$ (art. 34), and $dU = (y \times x^{-\frac{1}{2}} dx) + (Y \times \frac{1}{2} x^{-\frac{1}{2}} dx)$ (art. 37), we have, when x receives the special determination, $x = 4$ centimetres, $dy = \frac{1}{4} dx$, $Y = \frac{1}{2} dx$, and

$U = \left(2 \times \frac{dx}{2}\right) + \left(4 \times \frac{dx}{4}\right) = 2dx$. We may, consequently, by laying off bh'' , dh' , gh , each equal to dx , or unity, and $h''k'' = dy = \frac{1}{4} dx = \frac{1}{4}$, $h'k' = dY = \frac{1}{2} dx = \frac{1}{2}$, $hk = dU = 2$, and uniting respectively the extremities of y , Y and U with the points k'' , k' and k , obtain the lines bk'' tangent to abc at b , ak' tangent to ade at d , and gk tangent to agk at g , the line agk being its own tangent.

47. The hypothesis that curves are composed of indefinitely small rectilinear elements, and that they are, consequently, not curves, but polygons of an infinite number of infinitely small sides, is ingenious but unsatisfactory. For, let it be admitted that the hypothesis is true: then, x being the abscissa of a point of tangency, $x + dx$ (dx being regarded as infinitely small) will also be an abscissa of the same point of tangency, the infinitely small element of the curve, of whose extremities x and $x + dx$ are the abscissas, being nothing other than the point itself of tangency, but that point considered as possessing an infinitely small prolongation. By the hypothesis, therefore, $x + dx$ is the abscissa of a point situated, not only in the tangent, but also in the curve, since the infinitely small element of the tangent, of whose extremity $x + dx$ is the abscissa, is identical with an infinitely small element of the curve. Consequently, $y + dy$, the ordinate raised at the extremity of the abscissa $x + dx$, is the ordinate of a point situated as much in the curve as in the tangent; and since the point is thus situated in the curve, its ordinate may be computed from the equation of the curve. Let the equation of the curve be $U = y \times Y$, and, x receiving the infinitely small

increment dx , let y and Y , as ordinates of particular curves, receive the infinitely small increments dy and dY , we shall then have $U + dU = (y + dy) \times (Y + dY) = (y \times Y) + (y \times dY) + (Y \times dy) + (dy \times dY)$. But the presence of the term $dy \times dY$ refutes the hypothesis; since, according to the testimony of the advocates of the infinitesimal theory themselves, no such term ought to appear in the differential of $y \times Y$. And we cannot arbitrarily neglect this impertinent term, since logic, basing her prohibition on the axiom that that must be acknowledged to exist whose existence is adequately proved, forbids us to strike it out. For, in point of exact fact, since a curve and its tangent have the same direction at their point of tangency, and diverging directions on both sides of that point, and since, moreover, the divergency is continuously progressive, the angle made by the tangent and the curve being zero at the point of tangency, and becoming something on both sides of that point, it follows that the curve meets its tangent at an infinitely small angle, since that angle from something becomes *continuously* nothing when the curve falls into momentary coincidence with the tangent at their common point. Therefore, if x is the abscissa of the point of tangency, and dx an infinitely small quantity, the ordinate which corresponds to the abscissa $x + dx$; considered as ordinate to the tangent, will differ by an infinitely small quantity only from the ordinate which corresponds to the abscissa x ; because the trigonometrical tangent of a finite angle is infinitely small when measured upon an infinitely small circle. And since the angle made by the curve and its tangent is, in the immediate vicinity of the point of tangency, naturally infinitely small as compared with the angle made by the tangent and the axis of x , it follows that if we subject both angles to a scale of measurement the

gives the tangent of the finite angle as infinitely small, we shall have the tangent of the infinitely small angle (or, in this case, a proportionate function of that tangent) as infinitely smaller than the infinitely small tangent of the finite angle. Therefore, the ordinate corresponding to the abscissa $x + dx$, considered as ordinate to the curve, is equal to the ordinate corresponding to the abscissa x , plus the infinitely small trigonometrical tangent of the finite angle made by the tangent with the axis of x , that trigonometrical tangent measured to the infinitely small radius dx , plus or minus (according as the curve is convex or concave to the axis of x) a value which is naturally (in these circumstances of infinitely small variations resulting from the infinitely small variations of the abscissa) infinitely smaller than the infinitely small tangent represented by dU . Therefore, to reject the term $dy \times dY$, merely because it is infinitely small as compared with the infinitely small values dy and dY , is simply to refuse, in an arbitrary manner, to acknowledge the fact of curvature. Also, the refusal to regard lines as capable of being generated by the continuous motion of a point, that motion being itself subjected to a continuous change of direction, because such a conception is incompatible, on one hand, with the conception of curves as polygons made up of infinitely small rectilinear sides, and, on the other, with the theory that continuous increase takes place by mere accretions, is a sacrifice of comparatively clear ideas to ideas that are exceedingly obscure.

INVESTIGATION AND DETERMINATION OF THE NATURE AND OF THE ACCIDENTS OF CURVATURE.

Notation.

48. We obtain, by the differenciation of the two sides

of the general equation $y = f(x)$, the general expression $dy = f'(x) \times dx$, or, dividing by dx , $\frac{dy}{dx} = f'(x)$.

Representing $\frac{dy}{dx}$ by y' , we have $y' = f'(x)$; and since y' is, when so derived (except in a certain easily distinguishable particular case), itself a continuous function of x , nothing can prevent us from treating it as a function capable, in its own turn, of undergoing the process of differentiation. Differentiating, therefore, the two sides of the equation $y' = f'(x)$, we obtain $dy' = f''(x) \times dx$, or, dividing by dx , $\frac{dy'}{dx} = f''(x)$. Representing $\frac{dy'}{dx}$ by y'' , we have $y'' = f''(x)$, which expression is again (except in the like certain easily distinguishable particular case) itself capable of differentiation, etc.

49. In order to avoid a useless repetition of words, we shall, hereafter, call the general equation $y = f(x)$ (or the particular form which that general equation may take in particular cases, for a special example of which, see next article) *the primitive equation*; and the equation $y' = f'(x)$ — or the particular form, etc. — *the first derived equation*, or simply *the derived equation*. In like manner, we shall call the equation $y'' = f''(x)$, *the second derived equation*, and so on. We shall also characterise the successive functions as *primitive functions*, *first derived functions*, *second derived functions*, etc. Thus, y being the primitive function, y' , or $\frac{dy}{dx}$, or $f'(x)$, will be the derived function, or, more accurately, the *first derived function*; and y'' , or $\frac{dy'}{dx}$, or $f''(x)$, will be the *second*

derived function, etc. We shall also designate the line that has for its equation $y = f(x)$, as *the primitive line*; the line that has for its equation $y' = f'(x)$, as *the first derived line*, or simply *the derived line*; the line whose equation is $y'' = f''(x)$, as *the second derived line*, etc.

Of the successive derivation of lines.

50. We obtain, by the differentiation of the two sides of the particular equation $y = \frac{x^2}{2}$, the equation $dy = x \times dx$, or $\frac{dy}{dx} = x$. Putting $\frac{dy}{dx}$, or x , equal to y' and differentiating, we obtain $dy' = dx$, or $\frac{dy'}{dx} = 1$. In this particular case, therefore, we have y , or *the primitive function*, equal to $\frac{x^2}{2}$; y' , or *the first derived function*, equal to x ; and y'' , or *the second derived function*, equal to unity. Further than this we cannot go, since it is impossible, in this case, to continue the operation of derivation, because the constant quantity, unity, is incapable (art. 27) of being differentiated.

51. The second derived line, whose equation is $y'' = 1$, is (see fig. 5*) a right line drawn parallel to the axis of x , and at a distance from that axis equal to unity.

52. The first derived line, whose equation is $y' = x$,

* The first derived line is represented in the figure by a broken line with a single point in each of the breaks, thus: — . — . — ., and the second derived line is represented by a broken line with two points in each of the breaks, thus: — .. — .. — ..

is a right line cutting the axis of x at an angle of 45° (fig. 5), that is, at an angle whose trigonometrical tangent is y'' , or unity. For the ordinate of a line is nothing other than the trigonometrical tangent of the angle at which the tangent of the line, of which the line whose ordinate is considered is the first derived line, cuts the axis of x : thus, y'' being the ordinate considered, y'' is the trigonometrical tangent of the angle at which the line $y' = f'(x)$ cuts the axis of x ; for the line $y'' = 1$ is the first derived line of the line $y' = x$. Moreover, since, whatever may be the value of x , y'' is always equal to unity, the line $y' = x$ has, every where, the same inclination to the axis of x .

53. The primitive line, whose equation is $y = \frac{x^3}{3}$ (fig. 5), is — because y' the trigonometrical tangent of the angle at which its tangent cuts the axis of x is a regularly increasing function of x — itself subjected to a constant and regular change of direction, and therefore presents all the essential characteristics of curvature. The ordinate y has a regularly varying rate of increase, determined by the constant increase of the abscissa; and the function y has a regularly varying differential.

54. The consideration of the particular line $y = \frac{x^3}{3}$, and of its two derived lines $y' = x$ and $y'' = 1$, discloses a definition of curvature which, taking hold of the essence of things, reveals their hidden nature: *a curve is a line that has (at the least) a second derived line.*

This definition might, indeed, have been deduced from a general consideration of the fact of curvature, but not more authentically than from the consideration of the particular

case we have chosen : we do not know that it contains anything that will strike the reader with astonishment, but we present it, nevertheless, as containing the whole theory of curvature which we promised, on a previous page (art. 45), to explain. The definition is (essentially) by no means a new one, and we disclaim any intention of applying for a patent for its exclusive use.

55. Admitting the correctness of this definition (and we distinctly affirm that its denial involves an absolute denial of the possibility of any transcendental calculus whatever, as well that of Leibnitz as any other), it follows that it is impossible for any curve to have any very small (even infinitely small) elements, which are, themselves, right lines. For, since right lines, parallel to the axis of x , have no derived lines, and since right lines, making an angle with the axis of x , have derived lines which are straight and parallel to the axis of x , it follows that, if a curve has, at a point having the special abscissa x , a straight element parallel to the axis of x , there will be a discontinuity in the first derived line of that curve, at the point which has the same abscissa x ; also, if the curve has, at a point whose abscissa is x , a straight element whose direction makes an angle with that of the axis of x , then there will be a discontinuity in the second derived line at the point whose abscissa is x . And, if any curve should be entirely composed of infinitely small straight elements joined one at the end of another, then the first derived line, instead of being a real line, would be a series of disconnected points, situated at different distances from the axis of x , and the essential characteristic of the first derived line would be that of discontinuity. Now, the sole consideration of the special curve $y = \frac{x^2}{2}$ refutes all these

conclusions which follow from the infinitesimal hypothesis by way of inevitable consequence. For, when x grows continuously, the extremity of the ordinate y'' describes, continuously, the right line $y'' = 4$, and the extremity of the ordinate y' describes, with continuity, the right line $y' = x$, while the extremity of the ordinate y describes, with continuity, the parabolic curve $y = \frac{x^2}{2}$; for the continuity of the line $y'' = 4$, which is evident and requires no proof, involves, of necessity, the continuity of the lines $y' = x$, and $y = \frac{x^2}{2}$. For every derived function is always the exact rate of increase of the function of which it is the first derived function, provided, always, the two functions are taken for the same value of the abscissa.

56. We can understand and appreciate a man who rejects the calculus of fluxions (we are not speaking of the far inferior theory of limits), giving as a reason that it involves considerations of time and velocity, nay, that it involves the conception of velocities of velocities of velocities, etc., a task under which the most robust imagination may break down without dishonour; or who rejects it because it is, in reality, a *transcendental* calculus which introduces a chasm into the unity of pure mathematical science. We can, on the other hand, understand and appreciate a man who rejects the calculus of Lagrange, giving as a reason that it is no transcendental calculus at all, but merely a sublime algebra — the play of Hamlet with the part of the prince of Denmark left out. Again, the systems of Newton and Lagrange being both *demonstrably correct*, the largest liberty must be allowed when the question arises of a choice between the two. Between two systems equally

true, one may legitimately choose, basing the motive of choice on æsthetic considerations of symmetry, or on mere considerations of practical convenience; but we submit that, when the choice is between truth on one side, and demonstrable error on the other, no such freedom can be allowed.

AUGUSTE COMTE says of the infinitesimal calculus: « We are obliged to admit, with Lagrange, that the conception of Leibnitz is radically vicious in its logical relations. He himself declared the notion of infinitely small quantities to be a false idea: and it is in fact impossible to conceive of them clearly, though we may sometimes fancy that we do so.» Again, « Leibnitz himself failed to justify his conception, giving, when urged, an answer which represented it as a mere approximative calculus, the successive operations of which might, it is evident, admit an augmenting amount of error * ». To these extracts we add still another, although we are unable to see why a theory, invented and first published by Berkely, should always be attributed to Carnot, Carnot not being a man who requires, or who would himself have desired, to receive honour, or to be held responsible, for merit or demerit not his own. « It was Carnot (Berkeley) who showed that the infinitesimal method is founded on the principle of the necessary compensation of errors... Carnot's (Berkeley's) theory is doubtless more subtile than solid; *but it has no other radical logical vice than that of the infinitesimal theory itself*, of which it is, as it seems to me, the naturel development and general explanation: so that it must be adopted

* « We might produce numerous instances of such errors from the writings of modern computists. The most celebrated of these, *Leibnitz* and *John Bernoulli*, will furnish us with sufficient examples, from the single error of supposing infinitely small arcs absolutely to correspond with their chords, — etc.» BENJAMIN ROBINS. — *Dissertation on the discourse of fluxions*, art. 31.

as long as that method is directly employed ». To this testimony we add that of Berkeley, who says that the reasoning employed to prove the truth of the infinitesimal calculus is of a nature « that would not be allowed — *in theology!* »

It is true that, in a work published subsequently to the one from which the foregoing extracts are taken, Comte says, « an irrevocable reprobation, emanating systematically from the subjective synthesis, puts aside, as equally irrational and useless, every attempt to demonstrate, after a deductive manner, the fundamental law of the calculus of indirect relations »; but we are confident that the case is by no means so desperate as this illustrious author would have us imagine.

Rule for the signs of dy and of y' .

57. Since, in this tract, the abscissas are counted, for the sake of convenience, from zero to plus infinity, ~~and the contrary~~, when taken on the right hand side of the origin; and from zero to minus infinity, ~~and the contrary~~, when taken on the left hand side of the origin of coordinates, it follows that a function of the form $a \times -x$ must be regarded as an increasing function of x , and a function of the form $\frac{a}{-x}$ as a decreasing function of x ; the sign — signifying, in such cases, that x is taken on the left hand side of the origin.

58. The reader may easily convince himself, by the sole inspection of the figure (fig. 6), that, when y is an increasing function of x , y and dy have, necessarily, the same sign, and that, on the contrary, when y is a decreasing function of x , y and dy have, necessarily, contrary signs.

59. From which it follows that, since the sign of y' , that is of $\frac{dy}{dx}$, depends as much upon that of dx as upon that of dy , dy and y' will have the same sign when x is taken on the positive side of the origin of coordinates, and contrary signs when x is taken on the negative side of the origin, x and dx having always, of necessity, the same sign.

Of convexity and concavity.

60. When a curve $y = f(x)$ turns its convex side towards the axis of x , its tangent is situated between itself and that axis (arts 8 and 42); and the ordinate of the curve, which is identical, when the abscissa is that of the point of tangency, with the ordinate of the tangent, becomes greater than the ordinate of the tangent when the abscissa receives a value a little greater than that of the point of tangency; because, beyond the point of tangency, the curve is situated at a greater distance from the axis of x than is the tangent. Therefore, y' , or *the rate of increase* of the ordinate of the curve $y = f(x)$, *takes*, beyond the point of tangency, if y is an increasing function of x , a *value greater* than the one it possesses at that point; since, otherwise, the ordinate of the curve, which has, at the point of tangency, the same rate of increase with the ordinate of the tangent, could not take, beyond the point of tangency, greater values than those taken by the ordinate of the tangent. And, on the contrary, if y is a decreasing function of x , y' takes, upon leaving the point of tangency, a value less than the one it possesses at the point of tangency; for, otherwise, the curve $y = f(x)$ would not be situated beyond the point of tangency, at a greater distance from the axis of x than is

the tangent. — By the expression « beyond the point of tangency » is meant « for values of the abscissa a little greater than that of the point of tangency ».

61. When, therefore, a curve $y = f(x)$ is convex towards the axis of x , and y is an increasing function of x , y' is also an increasing function of x (art. 60), and y' and dy' have the same sign (art. 58). Consequently, if x and y are both positive, y , $\frac{dy}{dx}$, or y' , and $\frac{dy'}{dx}$, or y'' , will all have the same sign; if x is positive and y negative, then $-y$, $-\frac{dy}{dx}$, or $-y'$, and $-\frac{dy'}{dx}$ or $-y''$, will also all have the same sign; if x and y are both negative, then $-y$ and $-\frac{dy}{dx}$, or y' , will have contrary signs, and y' and $+\frac{dy'}{dx}$, or $-y''$, will also have contrary signs, so that, in this case as in the others, y and y'' will have the same sign; and finally, if y is positive and x negative, y and $+\frac{dy}{dx}$, or $-y'$, will have contrary signs, and $-y'$ and $-\frac{dy'}{dx}$, or y'' , will have contrary signs, so that, here also, y and y'' have the same sign. (See articles 57, 58 and 59.)

62. When, on the contrary, a curve $y = f(x)$ is convex towards the axis of x , and y is a decreasing function of x , y' is also a decreasing function of x (art. 60), and y' and dy' (art. 58) have contrary signs. Consequently, x and y being both positive, y and $-\frac{dy}{dx}$, or $-y'$, have contrary signs, while $-y'$ and $\frac{dy'}{dx}$, or y'' , have also contrary

signs, that is to say, y and y'' have the same sign ; if y is negative and x positive, $-y$ and $\frac{dy}{dx}$, or y' , have contrary signs, while y' and $\frac{-dy'}{+dx}$, or $-y''$, have also contrary signs, that is to say, y and y'' have the same sign ; if x and y are both negative, then $-y$ and $\frac{+dy}{-dx}$, or $-y'$, have the same sign, while $-y'$ and $\frac{+dy'}{-dx}$, or $-y''$, have also the same sign, that is to say, y and y'' have the same sign ; and finally, if y is positive and x negative, y and $\frac{-dy}{-dx}$, or y' , have the same sign, while y' and $\frac{-dy'}{-dx}$, or y'' , have also the same sign, that is to say, y and y'' have the same sign.

63. Therefore, in every case, x and y being either positive or negative, *if a curve $y = f(x)$ is CONVEX towards the axis of x , y and y'' have the same sign :*

And it may, in like manner, be shown that, in every case, x and y being either positive or negative, *if a curve $y = f(x)$ is CONCAVE towards the axis of x , y and y'' have contrary signs.*

Of greatest and least ordinates.

64. When a variable ordinate y increases continuously (with the continuous increase of x) until it attains a certain definite magnitude, and, that magnitude being once reached, commences continuously to decrease, then the ordinate in which the increase ends and the decrease begins is called *a maximum* (fig. 7).

65. And, conversely, when a variable ordinate y decreases continuously until it attains a certain definite magnitude, and, that magnitude being reached, commences, at once, to increase continuously, then the ordinate in which the decrease ends and the increase begins is called a *minimum* (fig. 8).

66. When the ordinate y , of the curve $y = f(x)$, is the ordinate of a point at which, if a tangent be drawn to the curve, that tangent will have an angle of inclination to the axis of x , it is evident, from the nature of the case, that such ordinate is neither a maximum nor a minimum; since, under such circumstances, we may always draw, on one side of y , an ordinate greater than y , and, on the other side, an ordinate less than y , the tendency of y to increase or decrease not being exhausted at the point whose coordinates are x and y . It follows, therefore, that the tangent which is drawn through the extremity of a maximum or a minimum ordinate, has a direction parallel to that of the axis of x .

67. Moreover, since y is (x continuously increasing) an increasing function of x when it is approaching the position where it has a maximum value (art. 64), and a decreasing function of x when it is receding from that position, it follows that a curve, having a maximum ordinate, is always *concave* towards the axis of x at the point at which the ordinate takes its maximum value; for the ordinates of the curve are, in such case, necessarily, on both sides of the maximum ordinate, of less value than the corresponding ordinates of the tangent. It may be shown, in like manner (although all this is perfectly evident from a mere inspection of the figures), that a curve having a minimum ordinate is

always *convex* towards the axis of x at the point where the ordinate takes its minimum value.

68. Therefore, when y is approximating to a maximum value, y is (art. 64) an increasing function of x ; consequently, y and dy have (art. 58), in such case, the same sign: moreover, when y attains its maximum value, dy (art. 23), and consequently y' , become equal to zero: finally, when y is receding from the position where it has a maximum value, y is (art. 64) a decreasing function of x ; consequently, y and dy have, in this case, contrary signs.

69. From which it follows that, when the ordinate y , of the curve $y = f(x)$, takes a maximum value, the first derived line $y' = f'(x)$ cuts the axis of x ; the ordinate y' of that line, changing its sign from $+$ to $-$, if x and y have the same sign, and from $-$ to $+$, if x and y have contrary signs (art. 59).

In like manner it may be shown that, when the ordinate y , of the curve $y = f(x)$, takes a minimum value, the first derived line $y' = f'(x)$ cuts the axis of x ; the ordinate y' , of that line, changing its sign from $-$ to $+$, if x and y have the same sign, and from $+$ to $-$, if x and y have contrary signs.

70. Example: If abc (fig. 40) be a curve whose equation is $y = \sqrt{2x - x^2}$, the first derived line will have for its equation $y' = \frac{1 - x}{\sqrt{2x - x^2}}$. Putting the numerator of this fraction equal to zero, in order to find the distance from the origin at which the first derived line cuts the axis of x , we have $x = 1$. Substituting for x , in the expression

$\frac{1-x}{\sqrt{2x-x^2}}$, values a little greater and a little less than unity, we find that, when the first derived line cuts the axis of x , y' changes its sign from $+$ to $-$; and that, consequently, $x = 1$ is an abscissa corresponding to a maximum ordinate (art. 69).

Putting x , the abscissa of the point of tangency, equal to zero, we have $y' = +\infty$: therefore, the axis of y is an asymptote of the first derived line. Putting $x = 2$, we have $y' = -\infty$: therefore, the line drawn tangent to the curve, at the point whose abscissa is equal to 2, is also an asymptote to the first derived line. Consequently, when x , commencing with the value $x = 0$, continuously increases, the point of tangency continuously advancing from a towards b , y' , commencing with the value $y' = +\infty$, continuously decreases until x takes the value $x = 1$, when y' takes the value $y' = 0$. Therefore, while x passes continuously through all the values that are comprised between 0 and 1, y' passes continuously through all the values that are comprised between $+\infty$ and 0. And, when x , commencing with the value $x = 1$, continuously increases, the point of tangency continuously advancing from b towards c , y' , commencing with the value $y' = 0$, continuously increases negatively, until x takes the value $x = 2$, when y' takes the value $y' = -\infty$. Therefore, while x passes continuously through all the values that are comprised between 1 and 2, y' passes continuously through all the values that are comprised between 0 and $-\infty$.

If the curve continually repeats itself, as in fig. 42, we shall have, at c , a cuspidate point. In this case, the primitive curve may, perhaps, be regarded as continuous, but since its direction is evidently discontinuous at the cus-

pidate points, the first derived line will necessarily be discontinuous.

Of contrary flexure.

71. A point of contrary flexure is one at which a curve, from being convex towards the axis of x , becomes concave; or, from being concave, becomes convex towards that axis.

72. When any curve, $y = f(x)$, has a point of contrary flexure whose special abscissa is x' , then the ordinate y'' , of its second derived line, necessarily (arts 74 and 63) undergoes a change of sign at the special point of the line $y'' = f''(x)$, whose abscissa is $x = x'$.

73. Examples: The curve whose equation is $y = \frac{x^3}{8} + \frac{x^2}{4} + x$ (figure 43) has for its first derived line the curve whose equation is $y' = \frac{1}{8}(3x^2 + 4x + 8)$, and for its second derived line the right line whose equation is $y'' = \frac{1}{8}(6x + 4)$. Putting the expression $6x + 4$ equal to zero, in order to discover at what distance from the origin of coordinates the line $y'' = \frac{1}{8}(6x + 4)$ cuts the axis of x , we obtain $x = -\frac{2}{3}$. Substituting in equation $y'' = \frac{1}{8}(6x + 4)$, for x , values a little greater and a little less than that of the special abscissa $x = -\frac{2}{3}$, we find that, when the second derived line cuts the axis of x , the sign of its ordinate changes from $+$ to $-$; consequently (art. 69),

the value $x = -\frac{2}{3}$ is the abscissa corresponding to a minimum ordinate of the first derived line $y' = \frac{1}{8}(3x^2 + 4x + 8)$.

Therefore, the value $x = -\frac{2}{3}$ is, at once, the distance at which the second derived line cuts the axis of x , the abscissa of a point at which the first derived line has a minimum ordinate, and the abscissa (art. 72) of a point at which the primitive line presents the phenomenon of contrary flexure. Moreover, since the ordinate y , of the point of contrary flexure, is negative, and since the ordinate y'' changes its sign from $+$ to $-$, when the line $y'' = f'(x)$ cuts the axis of x , it follows that (x increasing negatively) the primitive curve, from concave towards the axis of x , becomes convex towards that axis (art. 63) at the point whose abscissa is $x = -\frac{2}{3}$.

The curve, whose equation is $y = \left(\frac{x-1}{2}\right)^3 + 1$ (fig. 44), has for its first derived line the curve whose equation is $y' = \frac{3}{8}(x-1)^2$, and for its second derived line the right line whose equation is $y'' = \frac{3}{4}(x-1)$. Putting the expression $(x-1)$, which occurs in the equation of the second derived line, equal to 0, in order to discover at what distance from the origin that line cuts the axis of x , we obtain $x = 1$. Substituting, in equation $y' = \frac{3}{8}(x-1)^2$, for x , values a little greater and a little less than that of the special abscissa $x = 1$, we find that, when the second derived line cuts the axis of x , the sign of its ordinate changes (x increasing positively) from $-$ to $+$; conse-

quently (art. 69), the abscissa $x = 4$ corresponds to a minimum ordinate of the curve $y' = \frac{3}{8}(x - 4)^2$, and is also (art. 72) the abscissa of a point at which the curve $y = \left(\frac{x-4}{2}\right)^3 + 4$ has a contrary flexure; that curve becoming (x increasing positively) from concave towards the axis of x , convex towards that axis (art. 63), when x takes the value $x = 4$. Substituting, for x , in the equation of the first derived line, the special value, unity, we find that the ordinate of that line, corresponding to the abscissa $x = 4$, is $y' = 0$: but substituting, for x , in the equation $y' = \frac{3}{8}(x - 4)^2$, values a little greater and a little less than unity, we find that the first derived line touches the axis of x without cutting it, since, when it touches it, the ordinate y' undergoes no change of sign; and this is as it should be, because the abscissa $x = 4$ is the abscissa of a point of contrary flexure of the primitive line, and not the abscissa of a point where that line has a maximum or minimum ordinate.

The curve whose equation is $y = 2\sqrt[3]{(x-4)} + 4$ (fig. 45), has for its first derived line the curve whose equation is

$$y' = \frac{2}{3\sqrt[3]{(x-4)^2}}, \text{ and for its second derived line the curve}$$

$$\text{whose equation is } y'' = -\frac{4}{9\sqrt[3]{(x-4)^3}}. \text{ We cannot put}$$

$$\text{the numerator of the expression } -\frac{4}{9\sqrt[3]{(x-4)^3}}, \text{ equal to}$$

zero; it is, therefore, evident that the second derived line nowhere cuts the axis of x : but putting the denominator of that expression equal to zero, we obtain $x = 4$, and $y'' = \infty$. Moreover, substituting for x , in the equation of

the second derived line, values a little greater and a little less than the special value $x = 4$, we find that, when the abscissa takes the value $x = 4$, and the ordinate the value $y'' = \infty$, the ordinate y'' abruptly changes its sign from $+$ to $-$: the abscissa $x = 4$ is therefore the abscissa of a point of contrary flexure of the primitive curve. An inspection of the figure will suggest to the reader all that might be appropriately remarked in this place.

74. We omit the explanatory remarks that are promised in article 40, since we suppose the matter has now become too plain to require any further elucidation. Any one who prefers to do so is, of course, at liberty to regard a curve of contrary flexure as composed of two distinct branches that are tangent to each other at the point of contrary flexure, since, at that point, which the branches possess in common, the two have the same direction : and this simple observation may suffice to justify our definition.

Of Quadratures.

75. If, at different distances from the origin of coordinates, ordinates are raised to the special curve $y = f(x)$, the special spaces contained between the curve, the coordinate axes and the special ordinates (for example, the trilateral spaces obp, okq, fig. 4 ; obcp, fig. 5 ; ocbp, fig. 6 ; or the quadrilateral spaces oabp, fig. 7 and 44) will be special spaces, having definite values. By the expression, « the area of a curve », is meant, in these pages, the space thus included between the curve, the coordinate axes (or one only of these axes) and a special ordinate (figs 44 and 4), or between the curve and the coordinate axes only, as in the case of the space ocbp, fig. 6.

76. A curve being given by its equation $y = f(x)$, if we make x to vary continuously, y also will vary continuously, the extremity of y will continuously mark out a portion of the line $y = f(x)$, and the area, or space, of the line $y = f(x)$ will vary continuously; for, x changing its value by insensible gradations, the curve changes its direction by insensible gradations, and the area of the curve varies by insensible gradations. We here assume (as we suppose we are, by this time, justified in doing) that space (we do not say, matter) is infinitely divisible, and that the notion of ultimate infinitesimally small particles of space, which are themselves indivisible, is a notion to be rejected. When, therefore, x changes its value by a series of infinitely small variations (and it always thus changes its value when it is subjected to a finite *continuous* variation), then y also, and the area of the curve $y = f(x)$, change their values by undergoing a series of infinitely small variations.

77. If the line $y = f(x)$ be the special line $y = x^0$, or the line that is drawn parallel to the axis of x , and at a distance above the origin of coordinates equal to unity, then, when x , commencing at the origin, varies continuously, and from 0 becomes dx , the area of the line, also varying continuously, will from 0 become the square that has dx for its side. And since we have assumed $dx =$ one centimetre, as the unity of linear measure, we may now assume the square whose side is one centimetre, for the unity of superficial measure. We find, therefore, that, when x takes the special value, unity, the area under consideration takes also the special value, unity; and, in like manner, that, when x takes the special values 2, 3, 4, etc., the area takes the corresponding special values 2, 3, 4, etc., the number of the abscissa being, however, composed of

linear units, while that of the area is composed of superficial units. But if the line $y = f(x)$ is the special line $y = 2$, or the line that is drawn parallel to the axis of x at a distance from it equal to 2, then, when x from 0 becomes dx , the area of the line will from 0 become the rectangle which has for its base dx , and for its altitude 2 : if the line is the special line $y = 3$, then, when x from 0 becomes dx , the area of the line will become the rectangle whose base is dx , and whose altitude is 3, etc.

If the ordinate y have any numerical value whatever, n , then, when x varies continuously by the amount dx , the area of the line drawn parallel to the axis of x will vary continuously by the amount of the rectangle whose base is dx , and whose altitude is y , or n . And the rate of increase of the area of any line thus drawn whose ordinate is $n \times y$, is, therefore, n times as great as the rate of increase of the similar line whose ordinate is y . Consequently, *the number of the rate of increase of the area of a line drawn parallel to the axis of x , is nothing other than the numerical value of the ordinate of that line.*

78. Let, now, the line $y = f(x)$ be any right line whatever that is drawn at an angle of inclination to the axis of x ; and let it be required to determine *the rate of increase of its area*. The increase will, in this case, be variable, and the rate of increase will also be variable; but the rate of increase of the area of any line, for any special determination of x , is the rate of increase which the area has for that special determination of x , in contradistinction from what it has for other determinations of x . The rate of increase is, therefore, x receiving a special determination, not variable but *special*; and nothing other than what the rate of the actual increase would be if the

area should begin to increase *constantly* as well as continuously, as soon as the abscissa takes a value greater than x .

But, if the area begins to increase constantly as soon as x takes a value greater than the special determination x , it becomes identified, for values of the abscissa greater than x , with the area of the line drawn parallel to the axis of x , through the point whose special coordinates are x and y . Therefore, *the number of the rate of increase of the area* of the line $y = ax$ is, when the coordinates receive the special determination x and y , the number of the special ordinate y , of the line $y = ax$.

79. The foregoing reasoning holds good, not only when the equation $y = f(x)$ is the equation of a right line, but also when it is the equation of a curve; because, x receiving any determination whatever; if, for values of x beyond that special determination, the increase of the area is to be taken as constant, that area must be subjected to the condition mentioned in the preceding article. Whatever inclination, therefore, a line may have to the axis of x , be that inclination either constant or variable, if the variation of the area of that line, beyond a certain ordinate, is to be taken as constant, and as retaining always the same rate that it has when x takes the special determination x , then the increase of that area beyond that special ordinate must be regarded as identical with the increase of the area of the right line that is drawn parallel to the axis of x and through the extremity of that ordinate. Consequently, *the number of the rate of increase of the area* of any curve whatever is, when the coordinates receive the special determination x and y , nothing other than the number of the special ordinate y of that curve.

80. Let there be, now, a linear ordinate, z , the law of whose formation and growth shall be as follows : when x receives any determination whatever, as containing a particular number of linear units, and the area of the curve $y = f(x)$, receives a corresponding determination, as containing a correlative particular number of superficial units, then z shall also receive a special correlative determination, as containing a number of linear units equal to the number of superficial units contained (for that determination of x) in the area. In such case, if x vary continuously, the area will vary continuously, and z also will vary continuously, the extremity of z marking out a line the number of whose ordinate will always equal the number of the area of the line $y = f(x)$. *The analogy of the ordinate, z , to the area of the curve $y = f(x)$ will thus become perfect, and may, consequently, be made the basis of exact reasoning.* When, therefore, x receives the special determination x , z receives the special correlative determination, z , and the rate of increase of z receives the special determination $\frac{dz}{dx}$ (art. 26). But since the rate of the linear increase of z is identical with the rate of the superficial increase of the area, the number of the rate of increase of z is the number of the ordinate y , of the curve $y = f(x)$. If, therefore, we draw, through the point whose coordinates are x and z , a line parallel to the axis of x , and lay off on that line, counting from the point (x, z) , $dx = \text{unity}$, and then erect, at the extremity of dx , a line perpendicular to dx , laying off on that perpendicular, counting from the extremity of dx , a distance equal to y , and finally join the extremity of the line so laid off on the perpendicular with the point (x, z) , then the line joining these two points will be tangent to the line whose ordinate is z , at the point (x, z) , and we shall have

$$\frac{dz}{dx} = z' = y,$$

$$\text{or } dz = y \times dx.$$

81. By consequence, when we have a series of functions, y' , y'' , y''' , etc., each function being the first derived function of the one that immediately precedes it in the order of the series, it follows, not only that each function in the series represents the rate of increase of the function from which it is immediately derived, but also that it represents the area of the line of which the function, that is immediately derived from itself, is the ordinate. Thus we have, on one side, $y' = \frac{dy}{dx}$, and, on the other, $y' = \text{area}$ of curve $y'' = f''(x)$; also, $y'' = \frac{dy'}{dx}$, and $y'' = \text{area}$ of curve $y''' = f'''(x)$; and $y''' = \frac{dy''}{dx} = \text{area}$ of curve $y^{iv} = f^{iv}(x)$.

82. Example : If the line oa (fig. 47) be the curve whose equation is $y = \frac{x^3}{6}$, and the line ob the curve whose equation is $y' = \frac{x^2}{2}$, then the number of the ordinate of the line oa will be equal to the number of the correlative area of the line ob; in like manner, if oc be the line whose equation is $y'' = x$, the number of its area will be equal to the number of the ordinate of ob, while the number of its own ordinate will be equal to the number of the correlative area of the line de, whose equation is $y''' = 1$. Thus, when $x = 2$, we have $y = 4 + \frac{1}{3}$,

which is the number of the area of the line $y' = \frac{x^2}{2}$, for the special determination of x , $x=2$; we have also $y'=2$, which is the number of the area, for this special determination of x , of the line $y'' = x$: again, the equation $y'' = x$ gives us, when $x=2$, $y''=2$, which is the number of the area of the line $y''' = 1$, as taken for this special determination of x . If we wish to determine the number of the area of the primitive line $y = \frac{x^3}{6}$, for the value $x=2$, we must substitute, in the equation $Y = \frac{x^4}{24}$, which is the equation of the line of which the curve $y = \frac{x^3}{6}$ is the first derived line, instead of x , the value 2, thus obtaining $Y = \frac{2}{3}$, which is, at once, the linear number of the ordinate Y , and the superficial number of the area whose value is sought.

Observations.

83. Since it is more easy to explain the terms « law » and « analogy » than to define them, and as it has become a matter of some moment that the meaning of those terms should be made perfectly clear, the reader will permit us to adduce a series of illustrations : —

From the lily seed springs forth always the lily plant, and from other seed, other plants, according to their kind. The root, leaves, flowers, seeds, of the lily are always the root, leaves, flowers, seeds of the lily, and never of the rose or violet. Moses writes that, in the beginning, after having created the dry land, God said : « Let the earth

bring forth grass, the herb yielding seed, and the fruit-tree yielding fruit, *after his kind* ». If we take a twig from a peach-tree, and graft it into a plum-tree, the sap of the plum-tree will flow into the engrafted twig : the air, earth, water, that feed the tree will also feed the engrafted twig ; and the mere circumstances of the peach-twigs will be in all respects similar to those of the plum-twigs that are in its immediate vicinity. It would seem, therefore, that the peach-twigs ought to bear plums : nevertheless, it bears peaches, as it would have done if it had continued to live in its parent tree. The bark, fibres, leaves of the twig are always the bark, fibres, leaves of the peach-tree, and never of the plum. The twig remains always faithful to **THE LAW of its kind.**

84. Causes are not laws, neither are laws causes. We know little of the cause of gravitation, that is, of the mysterious force which brings gravitating bodies continuously and continually nearer to each other ; but we have no difficulty in stating one of its formulas or laws, namely that its intensity varies inversely as the squares of the distances. Law stands to cause in the same relation than form stands to substance ; and a law of a special cause is nothing other than one of the essential manners in which that special cause operates.

85. When we construct a curve upon paper, it is we, the living agents, who are the cause of the curve's existence ; and the manner in which we construct the curve is the law of the curve's existence. If we construct it by points determined by computation from the ordinary analytical equation of the curve, then we obtain a curve given in its different points. But a curve given in its different

points is a curve given under a point of view which excludes the consideration of continuity, and therefore no curve at all, but something altogether inadequate to its own nature or kind. It is by regarding lines as consisting in their different points that we fall between the two jaws of the antinomy which proceeds from the undeniable fact of the infinite divisibility (that is, of the continuity) of space. If, on the contrary, we construct the curve by a mechanical process, and by the continuous motion of a point, we obtain *the locus* of all the points determined by computation from the ordinary equation; and, when this is done (even in conception only), then the traitorous antinomy is shorn of its delusive powers, and its trick is laid bare. The ordinary equation is a law of the curve, but a law respecting the position of its points in relation to the coordinate axes; and it is, by no means, a law of the curve's formation and growth. The differential equation, on the other hand, is a law of the curve's formation and growth, since it has respect to the curve's different successive directions, and not at all to the position, but only to the direction of the coordinates axes.

86. It is possible that the signification we attribute to the word *law* is identical with the one which Plato attaches to the word *idea*; but if so, we must be careful to distinguish, with Plato and Parmenides, between *the idea* and *the ideal*: — the term, the ideal, being understood in the sense which the artists give it. For, if the law be a mere pattern or perfect model according to which things are made, developed or moulded, then things must resemble their ideas, and whenever we think of a thing and its law or idea, a new law, which is at once the law of the thing and of its ideal pattern, must rise up before

the mind. And, if we should consider this new law as a pattern, resembling the first law and the things formed in accordance with the first law, then another law would rise up before the mind, and so on to infinity : which shows the inconsistency of the hypothesis. — So much for the word *law* : we will now endeavour to illustrate the meaning we attach to the word *analogy*.

87. « It is unknown, at the present day », says Emanuel Swedenborg, in his treatise of Heaven and Hell, what *correspondence* is. This is owing to several causes, the chief of which is that man, has removed himself from heaven [the ideal?] through cherishing the love of self and of the world. For he who loves himself and the world above all things, looks to no other than worldly things, because they gratify the external senses and delight the natural disposition; and not to spiritual [ideal?] things, because these gratify internal senses and the internal mind. These, therefore, they reject, saying that they are too high to be objects of thought. The ancients did otherwise; for, to them, the science of correspondencies was the chief of all sciences... First then it shall be told what correspondence is. The whole natural world corresponds to the spiritual world; not only the natural world in general, but also in every particular; wherefore, whatever exists in the natural world from the spiritual is said to be correspondent. Every object in nature is said to be a correspondent in so far as it exists and subsists from Divine Order. »

We regret exceedingly that we are under the necessity of obscuring the brilliancy of Swedenborg's magnificent language by so commonplace an interpretation, but we feel bound to remark that what this spiritual writer calls *correspondency*, is (if we mistake not) what ordinary mortals

speak of as *analogy*. Indeed, there is a strong resemblance (some of our readers may regard it as a damaging one) between the theory of the Swedish seer as respects the relation of heaven to earth, and that of the substantial identity of the soul and the body of man, as taught by Spinoza. This resemblance appears more plainly in the following extract :

« Man is both a heaven and a world in miniature... Further, it is to be known that it is man by means of whom the natural world is conjoined with the spiritual, or that he is the medium of conjunction : for in him there is a natural world and there is a spiritual world. »

§§. Swedenborg says, moreover : « Though all things in heaven have their successions and progressions as in the world, the angels have no notion of time or space ; and, so completely destitute are they of such notion, that they do not even know what time and space are. In heaven, they have no years or days, but changes of state ; and where years and days exist, there are times and seasons, but where changes of state exist instead, there are states... Those are near each other who are in a similar state, and those are far apart whose state is dissimilar ; and spaces in heaven are nothing other than external states corresponding to internal ones... In the spiritual world, one person becomes present to another, provided, only, he intensely desires it ; for thus he views the other in thought, and puts himself in his state ; and one person is removed from another in the proportion that he holds him in aversion ; and as all aversions proceed from contrariety of affections and disagreement of thoughts, it hence results, that many who are there in one place so long as they agree, appear to each other ; whereas, as soon as they disagree, they disappear. Thus journeys are made in heaven, etc., etc. »

89. The value z , in the expressions, $z =$ ordinate, $z =$ area, is the representative of two distinct things that have an *analogy* to each other; for the ordinate and space to which z is equal, are identical to each other, inasmuch as they are both constituted by the same number, and different from each other inasmuch as they are constituted in different ways, the number in the two cases being differently embodied. But the equation is the same in both cases, because it is the number only of the quantities that can be identical, and therefore competent to balance itself when placed on both sides of the sign $=$; the quantities themselves being unlike in nature, and consequently refusing to submit to comparison — much more to equation. Thus, there is an analogy between the space and the ordinate, and that analogy is so strong that a single equation not only expresses the law of existence and development of the two, but also expresses it in such a way that every consequence drawn from the transformation of the equation, bears a twofold interpretation, one interpretation in accordance with the conditions of linear existence and applicable to the ordinate, and another interpretation in accordance with the conditions of superficial existence and applicable to the area. Every change, therefore, in the ordinate corresponds to an analogous change in the area, so that the two are adequate symbols (or, as Swedenborg would say, correspondencies) the one of the other.

90. We will illustrate these remarks by an example which will render the whole thing perfectly plain. Let it be required to find the greatest right angled triangle that can be constructed upon the hypotenuse 4 centimetres. Let x be one of the sides, and $z = \sqrt{46 - x^2}$ the

other. The area of the triangle will be, evidently, $area = \frac{x}{2} \sqrt{16 - x^2}$. So far all is clear; but if we make one step further in advance, we do it at the risk of plunging ourselves into thick darkness. Is the process of differentiation possible in this case? Tangents are drawn to lines, not to areas. What is there in common between contiguous areas and directions? It is lines, not areas, that have direction. How can a differential, which is a line constituting one side of the differential triangle, belong to a space? But these accumulating difficulties may all be avoided by a simple change in the form of the equation; for, if we write it $y = \frac{x}{2} \sqrt{16 - x^2}$, regarding y as an ordinate whose number is always identical with the number of a corresponding space, the expression $\frac{x}{2} \sqrt{16 - x^2}$ denoting, at one and the same time, the number of the ordinate and that of the space, according as the units are regarded as linear or superficial, the darkness vanishes. Differentiating both sides of the equation $y = \frac{x}{2} \sqrt{16 - x^2}$, and putting the expression for y' equal to zero, we obtain $x = \pm \sqrt{8}$ for the abscissa of the maximum ordinate. Substituting this value for x , in the equation $z = \sqrt{16 - x^2}$, we obtain $z = \pm \sqrt{8}$. Therefore, when x is equal to $\pm \sqrt{8}$, the ordinate, y , has a maximum value; and consequently, since the number of the ordinate, for every determination of x , corresponds to the number of the space, and since y , when x receives the determination mentioned, has attained a value where its increase ends and its decrease begins, the space also, when x receives that determination, has attained a value at which its increase ends and its de-

crease begins. Consequently, the triangle in question has the largest area when its two sides are equal, and each equal to many times one centimetre as there are units and parts of units in $\pm \sqrt{8}$. Thus, though it is absurd to speak of a tangent being drawn to a space, having the same direction with the space, and, withal, parallel to the axis of x , since a space is no line and has no direction, yet, by means of the principle of analogy, we may apply the calculus to questions of areas, questions to which, at first sight, it would appear to be altogether foreign by its nature. Yea, more, by means of the same principle, the calculus becomes applicable to questions of time, velocity, weight, heat, in short, to all questions implicating the consideration of continuity. If the reader thinks this recognition of the principle of analogy to be irregular, we would request him to reflect and to inquire whether any possible system of the calculus can escape the necessity of having recourse to analogy for the solution of problems? We contend that it is better to admit the principle openly, than to admit it under a disguised and bastard form.

91. When two things of different nature are subjected to one and the same *law*, the resemblance which subsists between them in consequence of that subjection, is called *analogy*.

92. When the law which constitutes the substance of the analogy is capable of an exact statement by means of one or more equations, the analogy is said to be mathematically *perfect*.

93. By means of perfect analogy we may know what takes place in one order of things by observing what takes

place in another order of things which is altogether distinct from the first; for, by a species of translation of one nature into a different nature, we may pass from the knowledge of the one, which is submitted to our direct observation, to the knowledge of the other whose modifications are hidden from our sight, and which we cannot approach otherwise than by the principle of analogy, or, if it be preferred, of exact correspondency. The first element, therefore, in the art of employing analogies consists in the subordinate art of distinguishing such analogies as are perfect, from such as are merely fantastical, — a task by no means difficult in mathematics.

THE THEOREM OF DEVELOPMENT.

Explanations and Notation.

94. The process which is the exact reverse of differentiation is called *integration*. To explain : if we have given a known differential, the process of integration will have for its object to find the function which, being differentiated, produces the given differential, and that function will be called the *integral* of the given differential. Although it is, in most cases, quite easy to find the differential of a given integral, it is sometimes extremely difficult, and perhaps impossible, to find the integrals of given differentials, especially when those differentials are not the actual result of exact differentiation. The treatment of the subject of integration neither comports with the limits which we have assigned for our exposition, nor with the nature of our work; since we have taken upon ourselves to explain, not practical and useful methods, but the nature

of curvature and the philosophy of the Calculus. We state, therefore, the following particular rule, which is the only one we shall probably have occasion to use, and then proceed with our subject :

95. *To find the integral of $mx^{m-1}dx$, increase the exponent of the variable by unity, divide the expression by the exponent thus increased and also by the differential of the variable.* — For it is obvious that, if the direct process gives us the differential of x^m equal to $mx^{m-1}dx$, the reverse process must give us the integral of $mx^{m-1}dx$ equal to x^m .

96. If $Y = F(x)$ be the equation of any line whatever, then $Y' = F'(x)$ will, in the subsequent pages of this sketch, stand for the equation of the line of which the line $Y = F(x)$ is the first derived line; and $Y'' = F''(x)$ will represent the equation of the line of which the line $Y = F(x)$ is the second derived line, and so on. Thus, if Y is any particular function of x , Y' is the function of which Y is the first derived function, Y'' is the function of which Y' is the second derived function, etc. In the series y, y', y'', y''' , etc.; we pass successively from one function to the function that is immediately derived from it, and from this last to the function that is immediately derived from this last, etc.; but in the series Y, Y', Y'', Y''' , etc., we pass successively from one function to the function *from which* the first is immediately derived, etc.; in the first case the transition taking place by the process of differentiation, while, in the second, it takes place by that of integration.

Further remarks on the succession of lines.

97. Let there be given the equation $Y = a$, which is the equation of a right line drawn parallel to the axis of x , and at a distance above that axis equal to a . To find the value of Y' , we must multiply a by dx , and take the integral of the resulting value, adx . For (art. 84) Y' is equal to the surface, or area, of the line $Y = a$, and the differential of Y' is equal to the differential of that area; we have, therefore (since the differential of that area is (art. 80) $a \times dx$), $dY' = adx$, or $\frac{dY'}{dx} = a$. But the integral Y' , of dY' , or of its equivalent adx , or of ax^0dx , is (art. 95) nothing other than ax : we have, therefore, $Y' = ax$. Y'' is equal (art. 84) to the area of the line; $Y' = ax$; dY'' is, therefore, equal to $axdx$, and, by consequence, Y'' is equal to the integral of $axdx$; we have, therefore, $Y'' = \frac{ax^2}{2}$. In like manner we find, $Y''' = \frac{ax^3}{2.3}$, $Y^{iv} = \frac{ax^4}{2.3.4}$, etc. And it results necessarily from the nature of the integration that, *at every step*, as we pass backward along the series $Y, Y', Y'',$ etc., the exponent of the variable x must increase by unity, while the denominator of the expression which contains that variable receives a new factor equal to the exponent so increased. We have therefore :

$$Y = a; \quad Y' = \frac{ax}{1}; \quad Y'' = \frac{ax^2}{1.2}; \quad Y''' = \frac{ax^3}{1.2.3};$$

A

$$\frac{1}{2.3.4} Y^{iv} = \frac{ax^4}{1.2.3.4}; \text{ etc.}$$

If now we make any one of these functions, for example $Y^{IV} = \frac{ax^4}{1.2.3.4}$, equal to y , we shall have a part of the same series under the reversed form :

$$y = \frac{ax^4}{1.2.3.4}; \quad y' = \frac{ax^3}{1.2.3}; \quad y'' = \frac{ax^2}{1.2}; \quad y''' = ax; \quad \text{B}$$

$$\text{and} \quad y^{IV} = a.$$

Comparing these two series, we find that we have :

$$Y = y^{IV}; \quad Y' = y'''; \quad Y'' = y''; \quad Y''' = y';$$

$$\text{and} \quad Y^{IV} = y. \quad \text{C}$$

98. [The equation $Y^{IV} = a \cdot \frac{x^4}{2.3.4}$ (or $y = a \cdot \frac{x^4}{2.3.4}$, since $Y^{IV} = y$), remains true in form, whatever may be the manner in which x is counted, provided the variation in the manner of counting violates none of the conditions under which the equation is obtained. If, therefore, we take for abscissa $x + h$, then $x + h$ will be a natural abscissa (but regarded in its parts and under a particular point of view) of the curve $y = a \cdot \frac{x^4}{2.3.4}$, and, in this case, we shall really consider two ordinates to the same curve, one corresponding to the abscissa x , the other to the abscissa $x + h$. Let U be a function of $x + h$, and such a function that U shall always be the ordinate of the curve $y = a \cdot \frac{x^4}{2.3.4}$, but taken for the abscissa $x + h$, instead of for the abscissa x . Then we shall have, $U = a \cdot \frac{(x + h)^4}{2.3.4}$;

a , the constant ordinate of the last derived line, being evidently in no way affected by the variations of the abscissa. Therefore,

$$U = \frac{a}{2.3.4} (x + h)^4 = \frac{a}{2.3.4} (x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4) \\ = a \left(\frac{x^4}{2.3.4} + \frac{x^3h}{2.3} + \frac{x^2h^2}{4} + \frac{xh^3}{2.3} + \frac{h^4}{2.3.4} \right),$$

and, consequently,

$$U - y = a \left(\frac{x^4}{2.3.4} + \frac{x^3h}{2.3} + \frac{x^2h^2}{4} + \frac{xh^3}{2.3} + \frac{h^4}{2.3.4} \right) - a \left(\frac{x^4}{2.3.4} \right) \\ = \frac{ax^3h}{2.3} + \frac{ax^2h^2}{4} + \frac{axh^3}{2.3} + \frac{ah^4}{2.3.4}.$$

Substituting, for the coefficients of the successive powers of h , their values as found in series B, we obtain,

$$U - y = y'.h + \frac{y''}{2}.h^2 + \frac{y'''}{2.3}.h^3 + \frac{y^{iv}}{2.3.4}.h^4;$$

U and y being two ordinates to the same line, one taken for the abscissa $x + h$, and the other for the abscissa x , or two areas of the first derived line of that line, one taken for the abscissa $x + h$, and the other for the abscissa x : the result which we should have obtained by means of Taylor's formula, if we had had recourse to it.

Observations.

99. We omit — since they will naturally suggest themselves to the reader — the observations that might appropriately be made in this place respecting the relations of curves to right lines, relations subsisting not between curves and imaginary small rectilinear elements of which those curves may hypothetically be regarded as composed, but relations subsisting between curves and their last derived lines; for last derived lines are always (when they actually exist) evidently right lines drawn parallel to the axis of x , It will be perceived that the relations to which we refer find the reason of their existence in the very nature and essence of curvature itself, implying that there is such a thing as real curvature, and that curves are actual curves, and not polygons with rectilinear sides. It is, therefore, not we who reject the consideration and notion of infinity, since we affirm that, regarding the element of a curve as something infinitely small, such infinitely small element itself, if it is anything, must necessarily present the phenomena of curvature.

100. It may be true that mathematical investigation lays a very slight tax on the higher faculties of the mind, but it must be confessed that it is, nevertheless, excessively tedious and wearing : nay, all the more tiresome because of its inferior character as a mental exercise. And for that reason, if for no other, we feel confident that we have not offended the reader by breaking off the reasoning contained in the immediately preceding section, adjourning its conclusions to a subsequent page.

The writer of this exposition has hardly looked into a mathematical book for twenty two years; he has, by consequence, forgotten the greater part of the little mathematical science he once possessed, and is often obliged, with extreme difficulty, to work up what he has to say from first principles : he therefore makes no secret of the fact that his interest in his labour depends, not on its mathematical character, but solely on the relation which the philosophy of the calculus holds to that of moral and social science.

101. We know not why it is so, but we can never think of the relation of metaphysics to actual, practical, definitively acquired (or, as we believe it is now called, *positive*) science, without thinking of the codfishery. Metaphysical truth presents itself to our imagination under the similitude of the living cod, swimming above the sand banks of the ree ocean, while definitively acquired truth is figured, or represented, by the image of the defunct cod, split open, salted and hung up by the shoulders in some encyclopædia or other back shop of constituted science. Far be it from us to express any sentiment of irreverence towards the deceased cod ! nevertheless, as a mere matter of taste, we had rather hunt the living fish, at the risk of being run down in the night by the American or English steamers, than to disembowel the dead one. — But the analogy is far from being perfect, and we accordingly refrain from presenting it as scientific.

Metaphysical research receives no fair play at the hands of the critics of the existing epoch. The metaphysician, when asked what he has to show as the result of his labours, produces the truth he has proved or explained ; but the scientific man objects that a truth which is actually

proved or explained, is one definitively acquired, and must be mentioned, if mentioned at all, in honour, not of metaphysics, but of constituted science. Thus unjustly is the mouth of the metaphysician completely and abruptly shut up. All the great improvements in science have been inaugurated by the metaphysicians, and resisted by the men of routine. Will the fact be denied? Take, by way of illustration, the very science upon which we are now engaged : was not Leibnitz, the inventor of the differential and integral calculus, a metaphysician by trade? was not Newton, the inventor of the calculus of fluxions and fluents, to say nothing of the inferior calculus of prime and ultimate ratios, a metaphysician of the very loftiest order? Much depends on the exact shade of meaning we attach to the term, « metaphysician ». We do not affirm that Newton never went astray ; for, evidently, placid mediocrity is the only thing that can be permanently relied on as not subject to aberration. We do not defend Newton for saying « attraction », when, perhaps, the facts of the case justified him in saying « gravitation » only ; neither will we attempt to defend his interpretation of the Apocalypse, since we never read that interpretation, and intend (if we do not change our mind) never to read it. Where there is a great fire, there is generally some smoke, and where there is no smoke at all, there is seldom any fire : besides, it is sometimes — especially under the broad light of the sun — difficult to distinguish between flame and smoke. Newton does not fall in our estimation because we are told that he took delight in reading the pages of Jacob Behmen ; but we contend that the fact (if authentic) proves him to have been no « positive » philosopher. If the scientific men see fit to reject Newton and send him ashore in the small boat, the metaphysicians will not fail to take him up.

102. We may adduce, in illustration of the methods of demonstration now in vogue, the most recent argument in proof of the infinitesimal calculus; an argument consisting in a historical verification of the fact that, in the interval between the revival of letters and time of Leibnitz, the notion of infinitesimally small quantities actually rose above the horizon of human thought, and gradually took upon itself a definitive form. What has this historical fact to do with the accuracy and truthfulness of the notion in question? Many things, since the beginning of the world, have risen and culminated. It is true that the argument may be — nay, actually has been — enriched by the historical verification of the correlative facts that, on one side, the infinitesimal notion dates back to the epoch of fetichism, and, on the other, is related to the hypothesis that forms the basis of the corpuscular theory. But, by a similar method of procedure, the practical or positive philosophy may prove anything, and everything, that is now, or ever has been, accepted by what is commonly called, « public opinion ». What has the verification of truth to do with the opinions of the promiscuous crowd of unthinking, or half-thinking, men of respectable standing, let that crowd be composed of marquises, pedants, coal-heavers, or even of grave philosophers? For grave philosophers, though ordinarily sedate and sober, not unfrequently, when under peculiar pressure, or concluding from uncertain and insufficient information, or anxious to give the last touch to a fantastical system, flash in the pan; and, in such case, their opinions ought evidently to count nothing. Against the vote of the majority, and of the representatives of the majority, an individual protest stands always good, provided it is grounded in the nature of things. To discover truth, the mind must remain open to the reception of the

particular light of truth, to the exclusion of all other light; and the first requisite of the philosophical method is (whatever the positive method may be) to remain always on one's guard against the influence of contemporary opinion and against that of scholastic prejudice — that is to say, to be on one's guard against the influence of the mere opinion of all classes and categories of men, be those men alive or dead. The sanction of the many confers, or confirms, authority, but cannot transform error into truth. What right have they who put their confidence in tradition and in accepted sentiments, to argue scornfully against the form of reasoning generally adopted in theology? It is hard for truth to contend against the prejudices of the present day and generation; but what will she be able to accomplish if the sanctity of the sepulchre is to be violated for her confusion, and the dead and buried opinions of past ages are to be disinterred and paraded as the genuine material of science? Give us the living and metaphysical codfish! It is said that metaphysical disquisitions are an amalgum of false science and bad poetry — with what justice we know not: but what shall we say of a historico-philosophical method which, regarded either as historical or philosophical, is false to its nature? Is there not a tradition from Galileo to Newton, as well as one from Cavalieri to Leibnitz? It is not enough that an argument is distinctly Hegelian: it ought also to be sound and conclusive. As for the historico-philosophical argument in favour of the infinitesimal notion, we have only to say, with the gentleman at the *table d'hôte*: Waiter, if this is tea, give us coffee, and if it is coffee, give us tea!

The Theorems of Maclaurin and Taylor.

103. Let mm' (fig. 48) be a curve whose equation is $y=f(x)$, and whose ordinate, at the point m , is om . Draw ms tangent to the curve at the point m , whose coordinates are x and y : draw also mr parallel to the axis of x , and passing through the point of tangency.

At a convenient distance from the ordinate om , taken at pleasure, say oo' equal to h , draw the ordinate $o'm'$. Join the extremities of the ordinates om , $o'm'$, by the secant line mm' . The coordinates of the point m' will be $x+h$ and U .

Let it be required to determine the value of the ordinate U , in terms of manageable functions of x and h .

104. It is evident, from inspection, that U , or $o'm'$, is equal to $o'r + rs + sm'$, that is equal to $y + y' \cdot h + \alpha \cdot h$; y' , or the tangent of the angle smr , taking the sign plus or the sign minus, according as y is an increasing or decreasing function of x , and α taking the sign plus or the sign minus, according as the curve $y = f(x)$ turns its convex or its concave side towards the axis of x . It may be shown, by a close examination of the nature of curvature, that $y \pm y' \cdot h \pm \alpha \cdot h$ is equal to $y \pm y' \cdot h \pm \frac{y''}{2} \cdot h^2 \pm \beta h$
 $= y \pm y' \cdot h \pm \frac{y''}{2} \cdot h^2 \pm \frac{y'''}{2 \cdot 3} \cdot h^3 \pm \gamma h$, and so on : but, since the process is somewhat tedious, we will endeavour to arrive at the same result in a shorter way.

105. Taking it for granted that the value of the ordinate $o'm'$, or U , is capable of being developed in a series of the form, $U = A + Bh + Ch^2 + Dh^3 + \dots$, (A, B, C , etc.

being constants, and it is admitted that our reasoning will lead to no useful result if it happen that the value of U is incapable of being so developed), we shall have,

$$U = A + Bh + Ch^2 + Dh^3 + \dots;$$

the coefficients A, B, C, D , etc. being altogether independent of the variations of h . Differentiating both sides of this equation, and dividing by dh , we obtain,

$$\frac{dU}{dh} = B + 2Ch + 3Dh^2 + \dots$$

Differentiating both sides of this last equation (regarding dh as constant, which we may properly do, since dh is a form of the differential of the independent variable), we obtain, after dividing by dh ,

$$\frac{d(dU)}{(dh)^2} = 2C + 2.3Dh + \dots$$

In like manner, we obtain,

$$\frac{d[d(dU)]}{(dh)^3} = 2.3D + \dots; \text{ etc.}$$

Now, since these equations are true for all values of h , they are true for the particular value $h = 0$.

Representing, therefore, by $[U]$, $\left[\frac{dU}{dh}\right]$, $\left[\frac{d(dU)}{(dh)^2}\right]$, etc., what U , $\frac{dU}{dh}$, $\frac{d(dU)}{(dh)^2}$, etc., become, when h becomes equal to zero, we have,

$$[U] = A; \left[\frac{dU}{dh}\right] = B; \left[\frac{d(dU)}{(dh)^2}\right] = 2C; \left[\frac{d[d(dU)]}{(dh)^3}\right] = 2.3D; \text{ etc.}$$

Therefore, by substituting, in the original equation the values thus found for A , B , C , etc., we obtain :

$$U = [U] + \left[\frac{dU}{dh}\right] \cdot h + \frac{1}{2} \left[\frac{d(dU)}{(dh)^2}\right] \cdot h^2 + \frac{1}{2 \cdot 3} \left[\frac{d[d(dU)]}{(dh)^3}\right] \cdot h^3 + \text{etc.}$$

106. Now, it must be noticed that $x + h$ is nothing other than the ordinary abscissa of the curve $y = f(x)$, but regarded under a particular point of view, and in its parts; and that U is nothing other than the ordinary ordinate of the curve, but regarded as taken for a particular form of the abscissa. Therefore, h is a particular form of the independant variable, regarded as varying in one only of its constituent parts; and the differential of h is the differential of the independant variable, that is to say, is the arbitrarily assumed value, unity; for, certainly, there is no reason for the variation of h , or for the determination of the rate of its variation, other than our own sovereign good pleasure. We have, consequently, $dh = dx$, a constant quantity entirely independant of the special determinations which x , h and $x + h$ may receive. Moreover, when h equals zero, $x + h$ equals x , and U equals y , that is, in such case, U , as is evident, is the ordinate of the curve $y = f(x)$, as taken for the abscissa x . We have, therefore,

$$[U] = y; \left[\frac{dU}{dh}\right] = \frac{dy}{dx} = y'; \left[\frac{d(dU)}{(dh)^2}\right] = \frac{d(dy)}{(dx)^2} = y''; \text{etc.}$$

Consequently,

$$U = y + \frac{y'}{1} \cdot h + \frac{y''}{1 \cdot 2} \cdot h^2 + \frac{y'''}{1 \cdot 2 \cdot 3} \cdot h^3 + \frac{y^{iv}}{1 \cdot 2 \cdot 3 \cdot 4} \cdot h^4 + \text{etc.}$$

And we may notice that, in this series, the signs of the alternate terms will be changed if h is negative, that the

implicit sign of the second term depends upon the character of the first term regarded as a decreasing or increasing function of x , that the implicit sign of the third term is determined by the fact of the convexity or concavity of the curve, and that the whole series becomes undermined in the very basis of its existence whenever either y' , y'' or....., becomes infinite, since, in such case, the differential triangle, of which dy , dy' or....., is the altitude, cannot be formed.

107. We may employ this series in all cases whatever, without misgivings as to the accuracy of the result, notwithstanding the arbitrary nature of the supposition with which we commenced its determination; for, first, if the foregoing reasoning is correct, the equation between U and the series must always be justly balanced; and, secondly, if the supposition that U can be developped in a series of the form $U = A + Bh + Ch^2 + \text{etc.}$, be totally unfounded, then the series as given by the formula

$$U = y + \frac{y'}{1}.h + \frac{y''}{1.2}.h^2 + \dots,$$

will become, not untrue (for that can never occur merely on such grounds), but unintelligible.

108. If the point, m , be situated in the axis of y , it will have for its coordinates $x = 0$, $y = (y)$; the brackets signifying that the ordinate is taken for the abscissa $x = 0$. In such case the series will present itself under the form

$$y = (y) + \left(\frac{y'}{1}\right)x + \left(\frac{y''}{1.2}\right)x^2 + \left(\frac{y'''}{1.2.3}\right)x^3 + \dots;$$

which is Maclaurin's Theorem; the brackets signifying that the values enclosed in them are taken for the particular value of the abscissa $x = 0$.

109. If the point m have for its coordinates x and y , the series will present itself under the form first determined (art. 106), y being the ordinate taken for the abscissa x , and U the ordinate taken for the abscissa $x + h$; as follows :

$$U = y + \frac{y'}{1}.h + \frac{y''}{1.2}.h^2 + \frac{y'''}{1.2.3}.h^3 + \dots;$$

which is Taylor's Theorem.

110. Examples : Let $y = (a + x)^m$: we shall have $y' = m(a + x)^{m-1}$; $y'' = m(m-1)(a + x)^{m-2}$; $y''' = m(m-1)(m-2)(a + x)^{m-3}$, etc., or, when x assumes the value $x = 0$, $(y) = a^m$, $(y') = ma^{m-1}$, $(y'') = m(m-1)a^{m-2}$, $(y''') = m(m-1)(m-2)a^{m-3}$, etc. Substituting these values for (y) , (y') , (y'') , etc., in Maclaurin's theorem, we obtain,

$$y = (a + x)^m = a^m + \frac{ma^{m-1}x}{1} + \frac{m(m-1)}{1.2}a^{m-2}x^2 + \frac{m(m-1)(m-2)}{1.2.3}a^{m-3}x^3 + \dots$$

or the binomial theorem. The theorem of Maclaurin comprehends, therefore, the binomial theorem.

In a similar manner it might be shown, if it were necessary, that the theorem of Taylor also comprehends the bi-

nomial theorem, since we may obtain, by its means, the series

$$(x + y)^n = x^n + nx^{n-1}y + \dots$$

Observations.

111. Berkeley makes, in the Analyst, for the purpose of refuting the hypothesis on which the Method of Limits is founded, the following statement, which we suppose may be admitted, and without discussion, as true :

« If with a view to demonstrate any proposition, a certain point is supposed, by virtue of which certain other points are attained; and such supposed point be itself afterwards rejected or destroyed by a contrary supposition; in that case, all the other points, attained thereby and consequent thereupon, must also be destroyed and rejected, so as from thence forward to be no more supposed or applied in the demonstration. »

112 The two members of the equation

$$\frac{U - y}{h} = y' + \frac{y''}{2} \cdot h + \frac{y'''}{2 \cdot 3} \cdot h^2 + \dots$$

(art. 409) differ in nature, the second being the more general of the two.

The expression $\frac{U - y}{h}$ denotes the tangent of the angle formed by the secant line mm' with the line mr , and is incapable of denoting anything else. Now, it will be noticed that, if h continuously decrease until it finally becomes zero, the point m' will run along the curve, in the direction from

m' towards m , until, when h becomes zero, m' will coincide with m , and the angle formed by the intersection of the secant mm' with mr will become identified with the angle formed by the intersection of the tangent ms with mr . In this case, the ordinate U becomes identified with the ordinate y , and we have $U - y = 0$; and this equality to nothing is not an equality to any mere approximative and fantastical zero, but an equality to nothing at all, the zero being of the most absolute kind. At the same time, the triangle mrm' passes utterly out of being, since its base, $mr = h$, becomes absolutely nothing, and its altitude, $rm' = U - y$, also absolutely nothing. When, therefore, $h = 0$, the expression $\frac{U - y}{h}$ takes the singular form $\frac{0}{0}$, a form implying, in this special case, not indetermination, but absurdity. And this ought to be so; for the expression $\frac{U - y}{h}$ (since it is constructed on the hypothesis that the triangle mrm' exists, and the whole root of its essence is grounded in that sole supposition) necessarily becomes utterly devoid of meaning when the conditions that give it a meaning cease utterly to be. The expression $\frac{0}{0}$ signifies, therefore, that the tangent of the angle mrm' has taken a value which the fraction $\frac{U - y}{h}$ is, by its nature, incompetent to express; that is, it signifies that the tangent of the angle formed with mr by mm' , has become the tangent of the angle formed with mr by the tangent to the curve, ms ; that is, again, it signifies that the secant mm' has passed utterly out of being, and that it is replaced by the tangent ms .

113. On the other hand, the second member of the equation, or $y' + \frac{y''}{2}.h + \frac{y'''}{2.3}.h^2 + \text{etc.}$, which is formed by differentiation, and therefore on a consideration as well of tangency as of secancy, does not become absurd when h becomes equal to zero, that is, when h receives the particular value which renders the fact of secancy impossible. Putting, therefore, in the equation

$$\frac{U - y}{h} = y' + \frac{y''}{2}.h + \frac{y'''}{2.3}.h^2 + \dots$$

h equal to zero, we obtain $\frac{0}{0} = y'$, or $\frac{0}{0}$ equal to the tangent of the angle smr : not $\frac{0}{0}$ equal generally to the tangent of the angle formed with mr by any secant whatever drawn through the point m , but $\frac{0}{0}$ equal to the tangent of the angle formed with mr by the one particular line which is no secant at all. Thus, the particular determination of h , $h = 0$, which, by the nature of things, renders the first member of the equation absurd, gives to the second member an intelligible value; and the real state and condition of the first member, when it is thrown into the singular form $\frac{0}{0}$, may, therefore, be inferred from the fact that the second member, with which it continues to remain equated, receives the contemporaneous determination, y' . We will, further on, illustrate the utility of these observations by an example.

DIFFERENTIATION OF TRANSCENDENTAL FUNCTIONS.

Of the rectification of curves.

114. When the rate of increase of the abscissa of a curve $y = f(x)$, is dx , and the rate of increase of its ordinate is dy , the rate of increase of the curve itself, (representing the curve itself, regarded under the point of view of its actual length, by z) will be $dz = \sqrt{(dx)^2 + (dy)^2}$. For when the increase of the ordinate from variable becomes constant, which is always the case when its differential is taken, then also the increase of the curve, as respects its length, from variable will become constant; because, in such case, the curve ceases to be a curve, and becomes identified with its tangent, whose increase is constant: therefore, the differential of a curve, that is to say, its rate of increase, at any point whose coordinates are x and y , is nothing other than the differential, or rate of increase, of the tangent that is drawn to the curve at that point. But the rate of increase of the tangent is, when the rate of increase of x is dx , and that of y , dy (as is evident from the nature of the differential triangle, which is right angled, having dx for its base and dy for its altitude), $\sqrt{(dx)^2 + (dy)^2}$. And this expression implies the nature of the curve, since the value of dy is derived, not from the equation to the tangent, but from the equation to the curve; the direction of the tangent line being determined, not independently, and from the consideration of the tangent itself considered as a particular right line, but from a consideration of the particular curve $y = f(x)$. Therefore, the value of z , the actual length of

the curve $y = f(x)$, may be determined by the integration of the expression $dz = \sqrt{(dx)^2 + (dy)^2}$; z being regarded as the ordinate of a curve $z = F(x)$ which has for the law of its growth that the rectilinear ordinate z , taken for the abscissa x , shall always be equal in length to the length which the curve $y = f(x)$ has at the point whose coordinates are x and y .

A right line drawn tangent to a circle is perpendicular to the radius that is drawn through the point of tangency.

115. Since the curvature of the circle is everywhere the same in the same circle, the circle departs by the same amounts, and in the same manner, on both sides of the point of tangency, from coincidence with its tangent: consequently, if a right line be drawn secant to a circle and parallel to its tangent, the two points of secancy, formed by the intersection of the right line with the curve, will be equally distant from the point of tangency. But since, by the nature of the curve, the two points of secancy thus determined are equally distant from the center, a perpendicular, let fall from the center upon the secant line, will bisect that part of it which joins the two points of secancy: consequently, that perpendicular, being prolonged, will pass through the point of tangency; and, since it is perpendicular to a line parallel to the tangent, it will be perpendicular to the tangent also; but since the perpendicular to the secant thus prolonged till it meets the tangent at the point of tangency, is nothing other than the radius, therefore, etc.

Differentiation of circular functions.

116. Let the arc ob (fig. 49) be denoted by z , and let b be the point of tangency : then by similar triangles, since dz , or the differential of the arc ob, is (art. 114) equal to bk , we have,

$$\begin{aligned} d \sin z : dz &:: \cos z : \text{unity}; \\ \text{or } d \sin z &= \cos z \, dz. \end{aligned}$$

$$\begin{aligned} \text{Also, } d \operatorname{versin} z : dz &:: \sin z : \text{unity}; \\ \text{or } d \operatorname{versin} z &= \sin z \, dz. \end{aligned}$$

$$\begin{aligned} \text{And, since } \operatorname{versin} z &= 1 - \cos z, \\ \text{or } d \operatorname{versin} z &= d(1 - \cos z) = -d \cos z, \\ \text{we have, } d \cos z &= -\sin z \, dz. \end{aligned}$$

$$\text{By consequence, since } \tan z = \frac{\sin z}{\cos z}$$

$$\text{we have, } d \tan z = \frac{dz}{(\cos z)^2}$$

and so on for other circular functions.

Differentiation of logarithmic functions.

117. Since $d(n \times y)$ is equal (art. 34) to $n \times dy$, it follows that the subtangent of a curve $Y = ny$, taken for the point whose coordinates are x and Y , is equal to the subtangent of its correlative curve $y = f(x)$, taken for the point whose coordinates are x and y . For, when the

ordinate y becomes ny , or Y , the differential of that ordinate becomes ndy , or dY : therefore, since (art. 24) the subtangent is equal to the ordinate divided by its differential and multiplied by the differential of the abscissa, the expression for the value of the subtangent undergoes no change when the ordinate becomes ny , and simply for the reason that the multiplication by n of the ordinate, which is found in the numerator of the expression for the subtangent, is accompanied by a simultaneous multiplication of its differential which is found in the denominator, by the same value n . Thus, the subtangent to the curve $y = f(x)$ is $\frac{f(x)}{d[f(x)]} \times dx$, and the subtangent to the curve $Y = nf(x)$ is $\frac{n \times f(x)}{n \times d[f(x)]} \times dx$, or precisely the same value.

118. Now, since logarithms are a series of numbers in arithmetical progression, answering, term for term, to another series of numbers which are in geometrical progression, it follows that, if the different values of x represent logarithms, and those of y their corresponding numbers, the ordinate of the logarithmic curve $y = f(x)$ will increase in geometrical ratio, while its abscissa increases in arithmetical ratio. If, therefore, when $x = 1$, $y = a$, and subtang. $= m$, then, when $x = 2$, y will become $n \times a$, and dy will become ndy , while the value of m will undergo no variation; for the expression which represents m remains unaffected by the change of y into ny , and of dy into ndy , since the only influence brought to bear upon it is the simultaneous multiplication of both its numerator and its denominator by the same number n . The subtangent of the logarithmic curve is, therefore, *constant*. But, by the nature of the differential triangle, we have

$$m : y :: d \log y : dy;$$

$$\text{or} \quad d \log y = m \frac{dy}{y}.$$

But this form of deduction, though perfectly in harmony with the principles set forth in the first and second chapter of this Sketch, is not the one which we prefer, on this occasion, to submit to the reader. Abandoning, therefore, the demonstration we have commenced, and, making an entire change in the method of our exposition, we shall take the liberty to determine the value of $d \log y$ by an altogether different process.

119. It being proposed to determine the differential of the quantity a^x , a being constant and x variable, we have, putting a^x equal to y , the equation, $y = a^x$, or the equation to a curve whose ordinate is the number of which its abscissa is the logarithm. If x become $x + \text{increment of } x$, y will become $a^{x + \text{increment of } x} = a^x \times a^{\text{inc } x} = U$.

To render this expression servicable, we will put $a = 1 + b$, and develop $a^{\text{inc } x}$, by developing its equivalent $(1 + b)^{\text{inc } x}$, by means of the binomial formula as deduced from *Maclaurin's theorem* (art. 110). We shall have,

$$\begin{aligned} a^{\text{inc } x} &= 1 + \text{inc } x.b + \text{inc } x \frac{\text{inc } x - 1}{1.2}.b^2 + \\ &\quad \text{inc } x \frac{(\text{inc } x - 1)(\text{inc } x - 2)}{1.2.3}.b^3 + \dots; \\ \text{or} \quad a^{\text{inc } x} - 1 &= \text{inc } x \left(b + \frac{\text{inc } x - 1}{1.2}.b^2 + \right. \\ &\quad \left. \frac{(\text{inc } x - 1)(\text{inc } x - 2)}{1.2.3}.b^3 + \dots \right), \end{aligned}$$

and, consequently,

$$\frac{U - y}{inc\ x} = \frac{(a^x \times a^{inc\ x}) - a^x}{inc\ x} = \frac{a^x(a^{inc\ x} - 1)}{inc\ x} = a^x \left(b + \frac{inc\ x - 1}{1.2} \cdot b^2 + \frac{(inc\ x - 1)(inc\ x - 2)}{1.2.3} \cdot b^3 + \dots \right).$$

120. The expression $\frac{U - y}{inc\ x}$ represents, or denotes, the tangent of the angle formed with the axis of x by the secant line which cuts the curve $y = a^x$ at the two points whose coordinates are, respectively, x and y , and $x + inc\ x$ and U ; U denoting the value $y + inc\ y$. When, therefore, we assign to $inc\ x$ the special value, $inc\ x = 0$, the fraction $\frac{U - y}{inc\ x}$, or $\frac{inc\ y}{inc\ x}$, assumes (art. 112) the singular form $\frac{0}{0}$, indicative of absurdity; since, when $inc\ x$ assumes that special value, the secant line passes utterly out of being, and is replaced by the tangent; while, under the same circumstances, the series (art. 113) with which $\frac{U - y}{inc\ x}$ is equated assumes a form which denotes the exact value of the tangent of the angle made by the intersection with the axis of x of the line that is drawn tangent to the curve at the point whose coordinates are x and y .

Putting, therefore, $inc\ x = 0$, we obtain,

$$\frac{0}{0} = a^x \left(b - \frac{1}{2}b^2 + \frac{1}{3}b^3 - \dots \right);$$

or, replacing b by its value $a - 1$,

$$\frac{0}{0} = a^x \left[(a-1) - \frac{1}{2}(a-1)^2 + \frac{1}{3}(a-1)^3 - \dots \right].$$

Moreover, since the second member of this last equation is equal to the trigonometrical tangent of the angle of intersection formed, etc., or, in short, equal to $\frac{dy}{dx}$, we obtain

$$\frac{dy}{dx} = a^x \left[(a-1) - \frac{1}{2}(a-1)^2 + \dots \right],$$

or, putting the factor $\left[(a-1) - \frac{1}{2}(a-1)^2 + \dots \right]$ equal to m' ,

$$\frac{dy}{dx} = m' a^x.$$

But, because (art. 449) x is the logarithm of y , dx is the differential of the logarithm of y ; we have, therefore,

$$dy = m' a^x d \log y;$$

or, denoting the fraction $\frac{1}{m'}$ by m ,

$$d \log y = \frac{m dy}{a^x} = m \cdot \frac{dy}{y}.$$

121. If, from the expression for m' , we calculate the value of a for the hypothesis $m=1$, it will be $e=2,71828$ nearly. The value m , which depends upon the value of the base a , of the system of logarithms employed, is called *the modulus* of the system.

122. Differentiating both sides of the equation to the curve $y = a^x$, we have

$$dy = \frac{y dx}{m}.$$

Substituting the value of $\frac{dx}{dy}$, as given by this last equation, in the general expression (art. 24) for the subtangent, we obtain

$$\text{Subtang.} = m,$$

that is to say, we find that the subtangent to the logarithmic curve is constant, and always *equal to the modulus of the system of logarithms from which the curve is constructed.*

123. In order to obtain a test for distinguishing those values of the area of a line which correspond to the ordinates of the curve from which that line is immediately derived, we must first find under what circumstances the number of some special space of the derived line becomes equal to the number of some ordinate of the primitive line. Thus the line $y'x = 1$ (which is an hyperbola referred to its asymptotes) being the first derived line of the logarithmic curve $e^y = x$, if we put $x = 1$, we shall have $y = 0$, $y' = 1$; in which case the number of the area of the curve $y'x = 1$, counted from the ordinate $y' = 1$ (that is, from the ordinate passing through the point at which the primitive line cuts the axis of x) will be equal to the number of the positive ordinate of the curve $e^y = x$; since, for values of the abscissa greater than $x = 1$, the ordinate of the primitive line, and the space of the derived line, increase correlatively,

simultaneously and equally. If the reader will draw various lines and the correlative derived lines, the inspection of the figures will suggest all that might be appropriately remarked in this place.

THE END.

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Fig. 19.

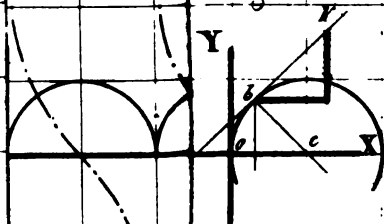


Fig. 10.

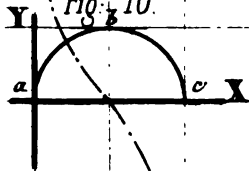


Fig. 18.

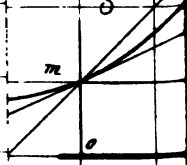


Fig. 5.

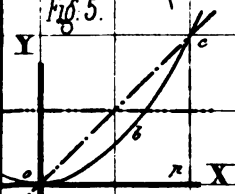
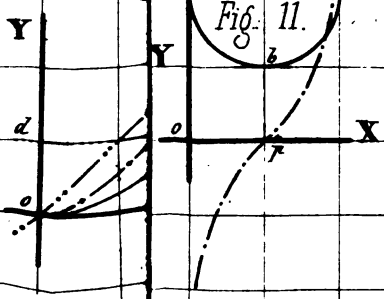
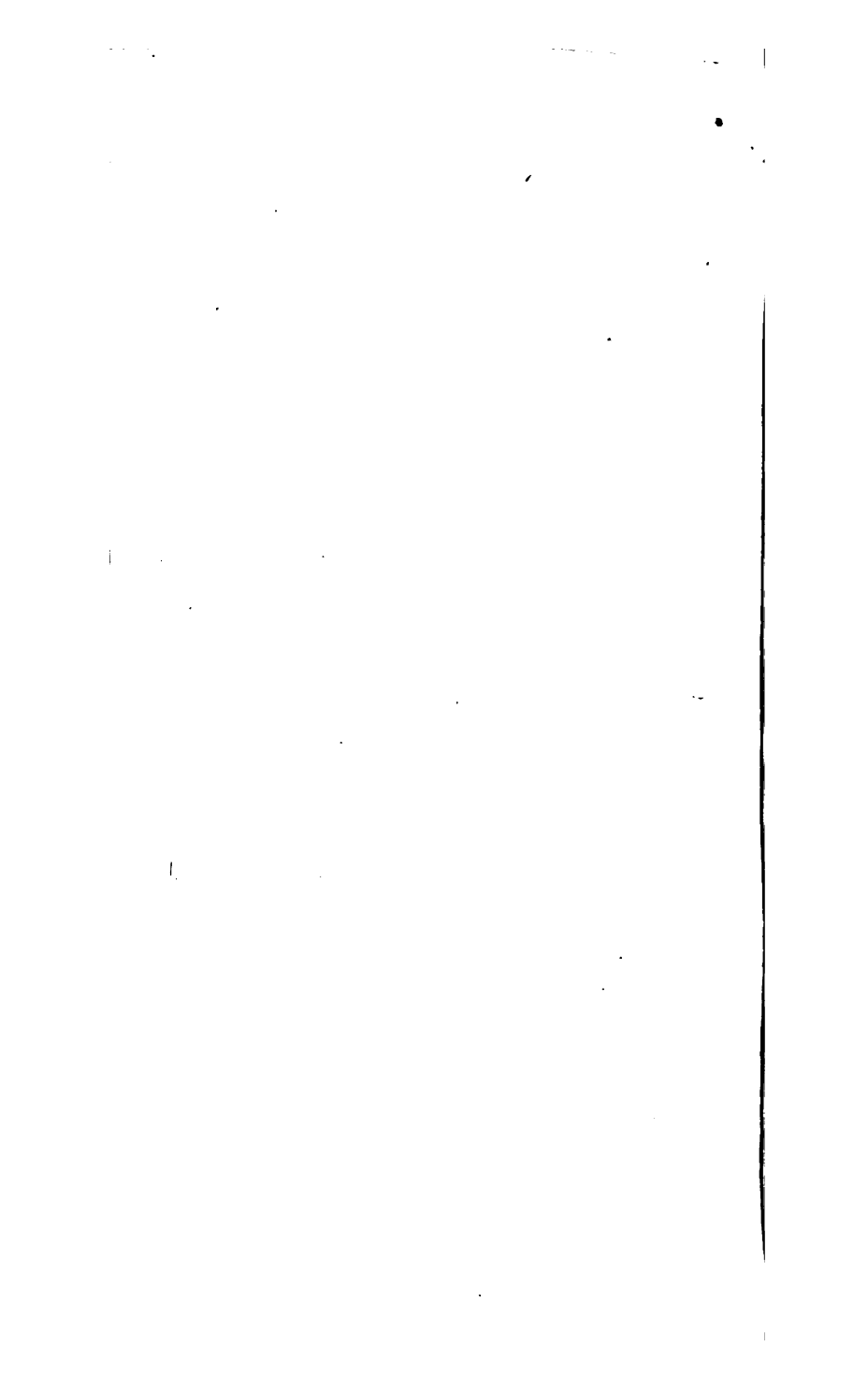
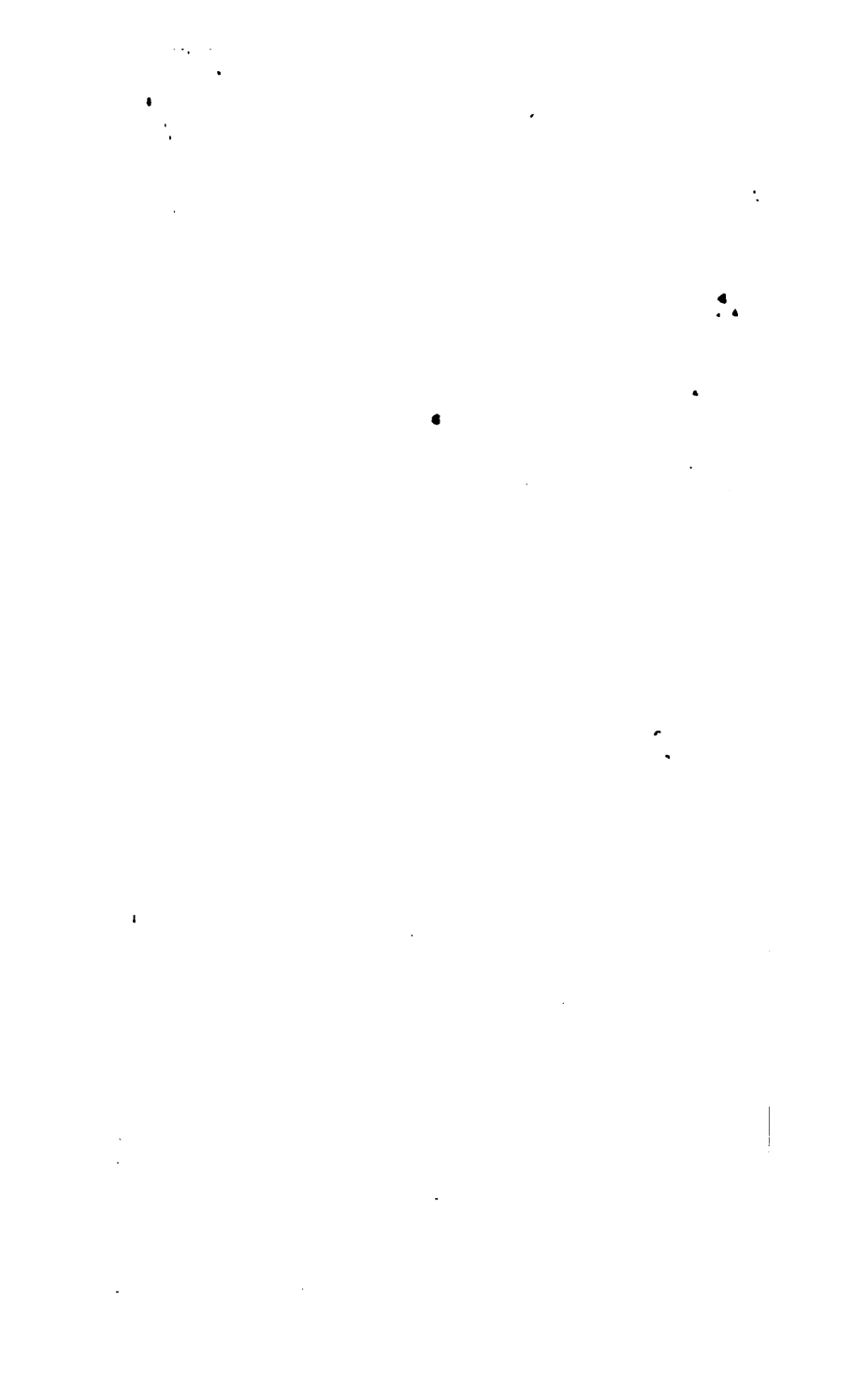


Fig.

Fig. 11.









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